Palestine Journal of Mathematics

Vol. 10(2)(2021) , 625–632

Gegenbauer Polynomials and Bi-univalent Functions

Ala Amourah, Adnan Ghazy Al Amoush and Mohammad Al-Kaseasbeh

Communicated by Faud Kittaneh

MSC 2010 Classifications: Primary 30C10, 30C45, 30C50 ; Secondary 33C45.

Keywords and phrases: Analytic functions, bi-univalent functions, Fekete-Szegö problem, Gegenbauer Polynomials, coefficient bounds, subordination.

Abstract. By means of Gegenbauer polynomials, two subclasses of analytic and bi-univalent functions are introduced. Coefficient bounds and Fekete-Szegö inequalities for functions belong to these subclasses are obtained.

1 Introduction

Orthogonal polynomials have been studied extensively as early as they were discovered by Legendre in 1784 [17]. In mathematical treatment of model problems, orthogonal polynomials arise often to find solutions of ordinary differential equations under certain conditions imposed by the model.

The importance of the orthogonal polynomials for the contemporary mathematics, as well as for wide range of their applications in the physics and engineering, is beyond any doubt. It is well-known that these polynomials play an essential role in problems of the approximation theory. They occur in the theory of differential and integral equations as well as in the mathematical statistics. Their applications in the quantum mechanics, scattering theory, automatic control, signal analysis and axially symmetric potential theory are also known [12, 13].

Formally speaking, polynomials P_n and P_m of order n and m are orthogonal if

$$\int_a^b w(x)P_n(x)P_m(x)dx = 0. \qquad \text{for} \qquad n \neq m,$$

where $w(x)$ is non-negative function in the interval (a, b); therefore, the integral is well-defined for all finite order polynomials $P_n(x)$.

A special case of orthogonal polynomials are Gegenbauer polynomials. They are representatively related with typically real functions T_R as discovered in [16], where the integral representation of typically real functions and generating function of Gegenbauer polynomials are using common algebraic expressions. Undoubtedly, this led to several useful inequalities appear from Gegenbauer polynomials realm.

Typically real functions play an important role in the geometric function theory because of the relation $T_R = \overline{co}S_R$ and its role of estimating coefficient bounds, where S_R denotes the class of univalent functions in the unit disk with real coefficients, and $\overline{co}S_R$ denotes the closed convex hull of S_R.

This paper associates certain bi-univalent functions with Gegenbauer polynomials and then explores some properties of the class in hand. Paving the way for mathematical notations and definitions, we provide the following section.

2 Preliminaries

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{2.1}$$

which are *analytic* in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Further, by S we shall denote the class of all functions in A which are *univalent* in $\mathbb{U}$.

A *subordination* between two analytic functions f and g is written as $f \prec g$. Conceptually, the analytic function f is subordinate to g if the image under g contains the image under f. Technically, the analytic function f is subordinate to g if there exists a Schwarz function w with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{U}$; such that

$$f(z) = g(w(z)).$$

Besides, if the function g is univalent in $\mathbb{U}$, then the following equivalence holds:

$$f(z) \prec g(z) \quad \text{if and only if} \quad f(0) = g(0)$$

and

$$f(\mathbb{U}) \subset g(\mathbb{U}).$$

Further on subordination principle we refer to [21].

The Koebe one-quarter theorem [14] asserts that the image of $\mathbb{U}$ under each univalent function f in S contains a disk of radius $1/4$. According to this, every function $f \in S$ has an *inverse map* f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}),$$

and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f); r_0(f) \geq \tfrac{1}{4}\right).$$

In fact, the inverse function is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{2.2}$$

A function $f \in A$ is said to be *bi-univalent* in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. let $\Im$ denote the class of bi-univalent functions in $\mathbb{U}$ given by (2.1). For early results on bi-univalent functions we refer to [18, 22].

For nonzero real constant α, a generating function of Gegenbauer polynomials is defined by

$$H_\alpha(x, z) = \frac{1}{(1 - 2xz + z^2)^\alpha}, \tag{2.3}$$

where $x \in [-1, 1]$ and $z \in \mathbb{U}$. For fixed x the function H_α is analytic in $\mathbb{U}$, so it can be expanded in a Taylor series as

$$H_\alpha(x, z) = \sum_{n=0}^{\infty} C_n^\alpha(x) z^n, \tag{2.4}$$

where $C_n^\alpha(x)$ is Gegenbauer polynomial of degree n.

Obviously, H_α generates nothing when $\alpha = 0$. Therefore, the generating function of the Gegenbauer polynomial is set to be

$$H_0(x, z) = 1 - \log\left(1 - 2xz + z^2\right) = \sum_{n=0}^{\infty} C_n^0(x) z^n \tag{2.5}$$

for $\alpha = 0$. Moreover, it is worth to mention that a normalization of α to be greater than $-1/2$ is desirable [13, 19, 24]. Gegenbauer polynomials can also be defined by the following recurrence relations:

$$C_n^\alpha(x) = \frac{1}{n}\left[2x\,(n + \alpha - 1)\, C_{n-1}^\alpha(x) - (n + 2\alpha - 2)\, C_{n-1}^\alpha(x)\right], \tag{2.6}$$

with the initial values

$$C_0^\alpha(x) = 1,\ C_1^\alpha(x) = 2\alpha x \text{ and } C_2^\alpha(x) = 2\alpha\,(1 + \alpha)\, x^2 - \alpha.$$

First off, we present some special cases of the polynomials $C_n^\alpha(x)$:

1. For $\alpha = 1$, we get the Chebyshev Polynomials.

2. For $\alpha = \frac{1}{2}$, we get the Legendre Polynomials.

Recently, many researchers have been exploring bi-univalent functions associated with orthogonal polynomials, few to mention [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 20, 23, 26, 25]. For Gegenbauer polynomial, as far as we know, there is no work associated with bi-univalent functions in the literatures. Initiating an exploration on properties of bi-univalent functions associated with Gegenbauer polynomials is the main goal of this paper. To do so, we take into account, the following definitions.

Definition 2.1 defines a class of convex bi-univalent functions associated with Gegenbauer polynomial as follows.

Definition 2.1. A function $f \in \Im$ given by (2.1) is said to be in the class $B_C(\alpha)$ if the following subordinations hold for all $z, w \in \mathbb{U}$:

$$1 + \frac{zf''(z)}{f'(z)} \prec H_\alpha(x, z) \tag{2.7}$$

and

$$1 + \frac{wg''(w)}{g'(w)} \prec H_\alpha(x, w), \tag{2.8}$$

where $x \in (\frac{1}{2}, 1]$, the function $g(w) = f^{-1}(w)$ is defined by (2.2) and H_α is the generating function of the Gegenbauer polynomial given by (2.3).

The following definition, defines a class of starlike bi-univalent functions associated with Gegenbauer polynomial.

Definition 2.2. A function $f \in \Im$ given by (2.1) is said to be in the class $B^*(\alpha)$ if the following subordinations hold for all $z, w \in \mathbb{U}$:

$$\frac{zf'(z)}{f(z)} \prec H_\alpha(x, z) \tag{2.9}$$

and

$$\frac{wg'(w)}{g(w)} \prec H_\alpha(x, w), \tag{2.10}$$

where $x \in (\frac{1}{2}, 1]$, the function $g(w) = f^{-1}(w)$ is defined by (2.2) and H_α is the generating function of the Gegenbauer polynomial given by (2.3).

3 Coefficient bounds for the function class $B_C(\alpha)$

This section is devoted to find initial coefficient bounds of the class $B_C(\alpha)$ of bi–univalent functions.

Theorem 3.1. *Let the function $f \in \Im$ given by (2.1) be in the class $B_C(\alpha)$. Then*

$$|a_2| \leq \frac{2\,|\alpha|\,x\sqrt{2\,|\alpha|\,x}}{\sqrt{|4\alpha\,(\alpha-1)\,x^2+2\alpha|}} \tag{3.1}$$

and

$$|a_3| \leq \alpha^2 x^2 + \frac{|\alpha|\,x}{3}. \tag{3.2}$$

Proof. Let $f \in B_C(\alpha)$. Then, Definition 2.1 allows the use of (2.7) and (2.8), and hence

$$1 + \frac{zf''(z)}{f'(z)} = H_\alpha(x, w(z)) \tag{3.3}$$

and

$$1 + \frac{wg''(w)}{g'(w)} = H_\alpha(x, v(w)), \tag{3.4}$$

for some analytic functions

$$w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots,$$

and

$$v(w) = d_1 w + d_2 w^2 + d_3 w^3 + \cdots,$$

on the unit disk $\mathbb{U}$ with $w(0) = v(0) = 0$, $|w(z)| < 1$ $(z \in \mathbb{U})$ and $|v(w)| < 1$ $(w \in \mathbb{U})$. By virtue of the generating function of the Gegenbauer polynomial H_α defined in (2.3), the equations (3.3) and (3.4), can be written as

$$1 + \frac{zf''(z)}{f'(z)} = 1 + C_1^\alpha(x) c_1 z + \left[C_1^\alpha(x) c_2 + C_2^\alpha(x) c_1^2\right] z^2 + \cdots$$

and

$$1 + \frac{wg''(w)}{g'(w)} = 1 + C_1^\alpha(x) d_1 w + \left[C_1^\alpha(x) d_2 + C_2^\alpha(x) d_1^2\right]) w^2 + \cdots.$$

A direct calculation shows that

$$2a_2 = C_1^\alpha(x) c_1, \tag{3.5}$$

$$6a_3 - 4a_2^2 = C_1^\alpha(x) c_2 + C_2^\alpha(x) c_1^2, \tag{3.6}$$

and

$$-2a_2 = C_1^\alpha(x) d_1, \tag{3.7}$$

$$8a_2^2 - 6a_3 = C_1^\alpha(x) d_2 + C_2^\alpha(x) d_1^2. \tag{3.8}$$

From (3.5) and (3.7), we have

$$c_1 = -d_1, \tag{3.9}$$

and

$$8a_2^2 = \left[C_1^\alpha(x)\right]^2 \left(c_1^2 + d_1^2\right). \tag{3.10}$$

Summing up (3.6) to (3.8), we get

$$4a_2^2 = C_1^\alpha(x) \left(c_2 + d_2\right) + C_2^\alpha(x) \left(c_1^2 + d_1^2\right). \tag{3.11}$$

By using (3.10) in (3.11), we get

$$\left[4 - \frac{8C_2^\alpha(x)}{\left[C_1^\alpha(x)\right]^2}\right] a_2^2 = C_1^\alpha(x) \left(c_2 + d_2\right). \tag{3.12}$$

It is well known that [14], if $|w(z)| < 1$ and $|v(w)| < 1$, then

$$|c_j| \leq 1 \text{ and } |d_j| \leq 1 \text{ for all } j \in \mathbb{N}. \tag{3.13}$$

By considering (2.6) and (3.13), we get from (3.12) the desired inequality (3.1).
Next, by subtracting (3.8) from (3.6), we have

$$12a_3 - 12a_2^2 = C_1^\alpha(x) \left(c_2 - d_2\right) + C_2^\alpha(x) \left(c_1^2 - d_1^2\right). \tag{3.14}$$

Further, in view of (3.9), it follows from (3.14) that

$$a_3 = a_2^2 + \frac{C_1^\alpha(x)}{12} \left(c_2 - d_2\right). \tag{3.15}$$

By considering (3.10) and (3.13), we get from (3.15) the desired inequality (3.2). This completes the proof of Theorem 3.1. □

Taking $\alpha = 1$ in Theorem 3.1, we get the following corollary.

Corollary 3.2. *Let the function $f \in \Im$ given by (2.1) be in the class $B_C(1)$. Then*

$$|a_2| \leq 2x\sqrt{x},$$

and

$$|a_3| \leq x^2 + \frac{x}{3}.$$

4 Coefficient bounds for the class $B^*(\alpha)$

This section is devoted to find initial coefficient bounds of the class $B^*(\alpha)$ of bi–univalent functions.

Theorem 4.1. *Let the function $f \in \Im$ given by (2.1) be in the class $B^*(\alpha)$. Then*

$$|a_2| \leq \frac{2\,|\alpha|\,x\sqrt{2\,|\alpha|\,x}}{\sqrt{|2\alpha\,(\alpha-1)\,x^2+\alpha|}} \tag{4.1}$$

and

$$|a_3| \leq 4\alpha^2 x^2 + |\alpha|\,x. \tag{4.2}$$

Proof. Let $f \in B^*(\alpha)$. From (2.9) and (2.10), we have

$$\frac{zf'(z)}{f(z)} = H_\alpha(x, w(z)) \tag{4.3}$$

and

$$\frac{wg'(w)}{g(w)} = H_\alpha(x, v(w)), \tag{4.4}$$

for some analytic functions

$$w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots \qquad (z \in \mathbb{U}),$$

and

$$v(w) = d_1 w + d_2 w^2 + d_3 w^3 + \cdots \quad (w \in \mathbb{U}),$$

such that $w(0) = v(0) = 0$, $|w(z)| < 1$ $(z \in \mathbb{U})$ and $|v(w)| < 1$ $(w \in \mathbb{U})$. It follows from (4.3) and (4.4) that

$$\frac{zf'(z)}{f(z)} = 1 + C_1^\alpha(x)c_1 z + \left[C_1^\alpha(x)c_2 + C_2^\alpha(x)c_1^2\right] z^2 + \cdots$$

and

$$\frac{wg'(w)}{g(w)} = 1 + C_1^\alpha(x)d_1 w + \left[C_1^\alpha(x)d_2 + C_2^\alpha(x)d_1^2\right])w^2 + \cdots .$$

A short calculation shows that

$$a_2 = C_1^\alpha(x)c_1, \tag{4.5}$$

$$2a_3 - a_2^2 = C_1^\alpha(x)c_2 + C_2^\alpha(x)c_1^2, \tag{4.6}$$

and

$$-a_2 = C_1^\alpha(x)d_1, \tag{4.7}$$

$$3a_2^2 - a_3 = C_1^\alpha(x)d_2 + C_2^\alpha(x)d_1^2. \tag{4.8}$$

From (4.5) and (4.7), we have

$$c_1 = -d_1, \tag{4.9}$$

and

$$2a_2^2 = \left[C_1^\alpha(x)\right]^2\left(c_1^2 + d_1^2\right). \tag{4.10}$$

By adding (4.6) to (4.8), we get

$$2a_2^2 = C_1^\alpha(x)\,(c_2 + d_2) + C_2^\alpha(x)\left(c_1^2 + d_1^2\right). \tag{4.11}$$

By using (4.10) in (4.11), we obtain

$$\left[2 - \frac{2C_2^\alpha(x)}{\left[C_1^\alpha(x)\right]^2}\right] a_2^2 = C_1^\alpha(x)\,(c_2 + d_2)\,. \tag{4.12}$$

Again, using the fact that if $|w(z)| < 1$ and $|v(w)| < 1$, then

$$|c_j| \leq 1 \text{ and } |d_j| \leq 1 \text{ for all } j \in \mathbb{N}. \tag{4.13}$$

By considering (2.6) and (4.13), we get from (4.12) the desired inequality (4.1).
Next, by subtracting (4.8) from (4.6), we have

$$4a_3 - 4a_2^2 = C_1^\alpha(x)\,(c_2 - d_2) + C_2^\alpha(x)\left(c_1^2 - d_1^2\right). \tag{4.14}$$

Further, in view of (4.9), it follows from (4.14) that

$$a_3 = a_2^2 + \frac{C_1^\alpha(x)}{4}\,(c_2 - d_2)\,. \tag{4.15}$$

By considering (4.10) and (4.13), we get from (4.15) the desired inequality (4.2). This completes the proof of Theorem 4.1. □

Taking $\alpha = 1$ in Theorem 4.1, we get the following corollary.

Corollary 4.2. *Let the function $f \in \Im$ given by (2.1) be in the class $B^*(1)$. Then*

$$|a_2| \leq 2x\sqrt{2x},$$

and

$$|a_3| \leq 4x^2 + x.$$

5 Fekete-Szegö inequality for the class $B_C(\alpha)$

Fekete-Szegö inequality is one of the famous problem related to coefficients of univalent analytic functions. It was first given by [15], who stated that, if $f \in S$, then

$$|a_3 - \eta a_2^2| \leq 1 + 2e^{-2\eta/(1-\mu)}. \tag{5.1}$$

This bound is sharp when η is real.

This section is devoted to find the sharp bounds of Fekete-Szegö functional $a_3 - \eta a_2^2$ for the class $B_C(\alpha)$ of bi–univalent functions.

Theorem 5.1. *Let the function $f \in \Im$ given by (2.1) be in the class $B_C(\alpha)$. Then for some $\eta \in \mathbb{R}$,*

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|\alpha| x}{3}, & |\eta - 1| \leq \left|\frac{1-2x^2}{6\alpha x^2}\right| \\[2ex] \frac{2\alpha^2 x^3 |1-\eta|}{|1-2x^2|}, & |\eta - 1| \geq \left|\frac{1-2x^2}{6\alpha x^2}\right|. \end{cases} \tag{5.2}$$

Proof. Let $f \in B_C(\alpha)$. By using (3.12) and (3.15) for some $\eta \in \mathbb{R}$, we get

$$\begin{aligned} a_3 - \eta a_2^2 &= (1-\eta)\left[\frac{\left[C_1^\alpha(x)\right]^3 (c_2 + d_2)}{4\left[C_1^\alpha(x)\right]^2 - 8C_2^\alpha(x)}\right] + \frac{C_1^\alpha(x)}{12}\,(c_2 - d_2) \\ &= C_1^\alpha(x)\left[\left(h(\eta) + \frac{1}{12}\right)c_2 + \left(h(\eta) - \frac{1}{12}\right)d_2\right], \end{aligned}$$

where

$$h(\eta) = \frac{\left[C_1^\alpha(x)\right]^2 (1-\eta)}{4\left[C_1^\alpha(x)\right]^2 - 8C_2^\alpha(x)}.$$

Then, we conclude that

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|\alpha| x}{3}, & |h(\eta)| \leq \frac{1}{12} \\ 4\,|\alpha|\,x|h(\eta)|, & |h(\eta)| \geq \frac{1}{12}. \end{cases}$$

This proves Theorem 5.1. □

Taking $\eta = 1$ in Theorem 5.1, we get the following corollary.

Corollary 5.2. *Let the function $f \in \Im$ given by (2.1) be in the class $B_C(\alpha)$. Then*

$$|a_3 - a_2^2| \leq \frac{|\alpha|\,x}{3}.$$

6 Fekete-Szegö inequality for the class $B^*(\alpha)$

Since the bounds of $|a_2|$ and $|a_3|$ are obtained for $f \in B^*(\alpha)$ in Section 4, then we are ready to find the sharp bounds of Fekete-Szegö functional $a_3 - \eta a_2^2$ defined for $f \in B^*(\alpha)$.

Theorem 6.1. *Let the function $f \in \Im$ given by (2.1) be in the class $B^*(\alpha)$. Then for some $\eta \in \mathbb{R}$,*

$$|a_3 - \eta a_2^2| \leq \begin{cases} |\alpha|\, x, & |\eta - 1| \leq \left|\frac{2\alpha x^2 - 2x^2 + 1}{2\alpha x^2}\right| \\ \frac{8|\alpha|^3 x^3 |1-\eta|}{|2\alpha(\alpha-1)x^2+\alpha|}, & |\eta - 1| \geq \left|\frac{2\alpha x^2 - 2x^2 + 1}{2\alpha x^2}\right|. \end{cases} \tag{6.1}$$

Proof. Let $f \in B^*(\alpha)$. By using (4.12) and (4.15) for some $\eta \in \mathbb{R}$, we get

$$a_3 - \eta a_2^2 = (1-\eta)\left[\frac{\left[C_1^\alpha(x)\right]^3 (c_2 + d_2)}{2\left[C_1^\alpha(x)\right]^2 - 2C_2^\alpha(x)}\right] + \frac{C_1^\alpha(x)}{4}(c_2 - d_2)$$

$$= C_1^\alpha(x)\left[\left(h(\eta) + \frac{1}{4}\right)c_2 + \left(h(\eta) - \frac{1}{4}\right)d_2\right],$$

where

$$h(\eta) = \frac{\left[C_1^\alpha(x)\right]^2 (1-\eta)}{2\left[C_1^\alpha(x)\right]^2 - 2C_2^\alpha(x)}.$$

Then, we easily conclude that

$$|a_3 - \eta a_2^2| \leq \begin{cases} |\alpha|\, x, & |h(\eta)| \leq \frac{1}{4} \\ 4\,|\alpha|\, x|h(\eta)|x, & |h(\eta)| \geq \frac{1}{4}. \end{cases}$$

This proves Theorem 6.1. □

Taking $\eta = 1$ in Theorem 6.1, we get the following corollary.

Corollary 6.2. *Let the function $f \in \Im$ given by (2.1) be in the class $B^*(\alpha)$. Then*

$$|a_3 - a_2^2| \leq |\alpha|\, x.$$

7 Conclusion

This research paper has introduced two subclasses of bi-univalent functions by means of Gegenbauer polynomials. For these subclasses, some properties have been investigated; namely, coefficient bounds and Fekete-Szegö inequalities.

References

[1] A. AlAmoush, Certain subclasses of bi-univalent functions involving the Poisson distribution associated with Horadam polynomials, *Malaya Journal of Matematik*, 7(4, 2019), 618-624 (2019).

[2] A. Alamoush, Coefficient estimates for a new subclasses of λ–pseudo biunivalent functions with respect to symmetrical points associated with the Horadam Polynomials, *Turkish Journal of Mathematics*, 43(6), 2865-2875 (2019).

[3] A. Alamoush, A subclass of pseudo-type meromorphic bi-univalent functions, *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, 69(2), 31-38 (2020).

[4] A. Alamoush, and M. Darus, On coefficient estimates for bi-univalent functions of Fox-Wright functions, *Far East J. Math. Sci.*, (89), 249–262 (2014).

[5] A. Amourah, Initial bounds for analytic and bi-univalent functions by means of (p, q)–Chebyshev polynomials defined by differential operator, *General Letters in Mathematics*, 7(2), 45-51 (2019).

[6] A. Amourah, Fekete-Szego inequality for analytic and bi-univalent functions subordinate to (p, q)–Lucas polynomials. *TWMS Journal of Applied and Engineering Mathematics*,(Accpted).

[7] A. Amourah, and M. Illafe, A Comprehensive subclass of analytic and bi-univalent functions associated with subordination, *Palestine Journal of Mathematics*,9(1), 187–193 (2020).

[8] A. Amourah, Faber polynomial coefficient estimates for a class of analytic bi-univalent functions, *AIP Conference Proceedings*,2096(1), 020024 (2019).

[9] A. Aljarah, Maymoona, and M. Darus, Neighborhoods for subclasses of p-valent functions defined by a new generalized differenitial operator, *ROMAI Journal*,15(1)(2019).

[10] B. Frasin, and MK. Aouf, New subclasses of bi-univalent functions, *Applied Mathematics Letters*, 24(9), 1569–1573 (2011).

[11] B. Frasin, Coefficient bounds for certain classes of bi-univalent functions, *Hacettepe Journal of Mathematics and Statistics*, 43(3), 383–389 (2014).

[12] H. Bateman, *Higher Transcendental Functions*, McGraw-Hill, 1953.

[13] B. Doman, *The classical orthogonal polynomials*, World Scientific, 2015.

[14] P. Duren, *Univalent functions*, Grundlehren der Mathematischen Wissenschaften 259, Springer-Verlag, New York, 1983.

[15] M. Fekete, and G. Szegö, Eine Bemerkung Ã$\frac{1}{4}$ber ungerade schlichte Funktionen. *Journal of the London Mathematical Society*, 1(2), 85-89 (1933).

[16] K. Kiepiela, I. Naraniecka, and J. Szynal, The Gegenbauer polynomials and typically real functions, *Journal of computational and applied mathematics*, 153(1-2), 273-282 (2003).

[17] A. Legendre, Recherches sur lattraction des spheroides homogenes, *Memoires presentes par divers savants al Academie des Sciences de lInstitut de France, Paris*, 10, 411-434 (1785) (Read 1784).

[18] M. Lewin, On a coefficient problem for bi-univalent functions, *Proceedings of the American Mathematical Society*, 18(1), 63-68 (1967).

[19] W. Liu, and L. Wang, Asymptotics of the generalized Gegenbauer functions of fractional degree. *Journal of Approximation Theory*, 253, 105378 (2020).

[20] N. Magesh, and S. Bulut, Chebyshev polynomial coefficient estimates for a class of analytic bi-univalent functions related to pseudo-starlike functions. *Afrika Matematika*, 29(1-2), 203-209 (2018).

[21] S. Miller and P. Mocanu, *Differential Subordination: theory and applications*, CRC Press, New York, 2000.

[22] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, *Archive for Rational Mechanics and Analysis*, 32(2), 100-112 (1969).

[23] H. Orhan, N. Magesh, and V. Balaji, Second Hankel determinant for certain class of bi-univalent functions defined by Chebyshev polynomials, *Asian-European Journal of Mathematics*, 12(02), 1950017 (2019).

[24] M. Reimer, *Multivariate polynomial approximation*. BirkhÃ¤user, 2012.

[25] F. Yousef, T. Al-Hawary, and G. Murugusundaramoorthy, FeketeSzego functional problems for some subclasses of bi-univalent functions defined by Frasin differential operator. *Afrika Matematika*, 30(3-4), 495-503 (2019).

[26] F. Yousef, B. Frasin, and T. Al-Hawary, FeketeSzego Inequality for Analytic and Bi-univalent Functions Subordinate to Chebyshev Polynomials, *Filomat*, 32(9), 3229–3236 (2018).

Author information

Ala Amourah, Department of Mathematics, Faculty of Science and Technology, The Irbid National University, Irbid, Jordan.
E-mail: alaammour@yahoo.com (Corresponding author)

Adnan Ghazy Al Amoush, Faculty of Science, Taibah University, Saudi Arabia.
E-mail: adnan_omoush@yahoo.com

Mohammad Al-Kaseasbeh, Department of Mathematics, Faculty of Science, Jerash University, Jerash, Jordan.
E-mail: m.kasasbeh@jpu.edu.jo

Received: September 3, 2020.
Accepted: November 29, 2020