

On Cyclic Triple System and Factorization

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Abstract: A near- k -factor of a graph G is a spanning subgraph in which exactly one isolated vertex and all other vertices of order k . In this study, we employ the near-four-factor concept and (m_1, m_2, \dots, m_r) -cycle system to present a new method for constructing a cyclic 12-fold triple system. Firstly, we would like to propose a new type of cyclic triple system called cyclic triple near factorization, denoted by $CTNF(v)$. Then, we prove the existence of $CTNF(v)$ along with an algorithm for starter triples of $CTNF(v)$ for $v = 12n+2$ when, n is even. Finally, we use the construction of $CTNF(v)$ to demonstrate the existence of $[a, b]$ factorization of $12K_v$ for $a = 8$ and $b = 4(v-1)$.

Key words: Triple system, near- k -factor, complete multigraph, starter triples, demonstrate, factorization

INTRODUCTION

All graphs under consideration are finite, undirected and without loop. An m -cycle, written $C_m = (c_0, \dots, c_{m-1})$, consists of m distinct vertices $\{c_0, c_1, \dots, c_{m-1}\}$ and m edges $\{c_i c_{i+1}\}$, $0 \leq i \leq m-2$ and $\{c_0 c_{m-1}\}$. An (m_1, m_2, \dots, m_r) -cycle system of graph G is a pair (V, C) where, V is the vertex set of G and C is the collection of $\{C_m, 1 \leq i \leq r\}$ for whose edge sets partition of the edge set of G and is called to be a cyclic if $V = Z_v$ and we have $C_{m+1} \in C$ whenever $C_m \in C$. In particular, if $m_1 = \dots = m_r = m$, it is called cyclic m -cycle system (Wu and Lee, 2008).

Let $d_G(v)$ be a degree of v in G . A graph G is called $[a, b]$ -graph if $a \leq d_G(v) \leq b$ for every $v \in V(G)$ and $b \geq a \geq 0$. An $[a, b]$ -factor of G is a spanning $[a, b]$ -subgraph and $[k, k]$ -factor is abbreviated to an k -factor. While a near- k -factor is a spanning subgraph in which all but one vertex has a degree k with the remaining vertex having degree 0 that is said isolated vertex. The decomposition of edge set of G into $[a, b]$ -factors (respectively, near- k -factor) is called a $[a, b]$ -factorization (respectively, near- k -factorization) (Mao-Cheng, 1991).

A balanced incomplete block design, denoted by (v, k, λ) -BIBD is a pair (V, B) where, V is a finite set of v points and B is a collection of k -subsets of V such that each pair of distinct points of V is contained in precisely λ blocks. It easy to see that any (v, k, λ) -BIBD can be viewed as a decomposition of complete multigraph λK_v , the graph with v vertices in which every two vertices are joined by λ parallel edges into copies of K_k . An

automorphism of BIBD (V, B) is a permutation τ on V such that $B = \{b_1, \dots, b_k\} \in B$ if and only if $\tau(B) = \{\tau b_1, \dots, \tau b_k\} \in B$. If there is an automorphism τ of order v , then, BIBD is called cyclic. Thus, the automorphism will be represented by $\tau: (0, 1, \dots, v-1)$. The orbit of the block B , denoted by $\text{orb}(B)$ is the set of all distinct blocks in the collection $\{B+i | i \in Z_v\}$. The orbit of block B is said to be a full if $\text{orb}(B) = v$ and otherwise is said a short (Mathon, 1987). A cyclic λ -fold triple system of order v , denoted by $CTS(v, \lambda)$ is cyclic $(v, 3, \lambda)$ -BIBD. A $CTS(v, \lambda)$ is simple, if it contains no repeated triple. When $v \not\equiv 0 \pmod{3}$ then, there is no short orbit of block (Tian and Wei, 2013).

The existence of $CTS(v, \lambda)$ for $v \equiv 1, 3 \pmod{6}$ have been studied by Colbourn and Colbourn (1981). While Colbourn and Rosa (1999) have given the necessary conditions for the existence of $CTS(v, \lambda)$. The existence of $CTS(v, \lambda)$ for any possible parameters v and λ is an interesting problem, since, this kind of design has a nice combinatorics and algebraic properties and also has a connection to the optical orthogonal codes (Chen and Wei, 2012).

Motivated by construction these designs, Tian and Wei (2010) provided a new method for decomposing all the triples of Z_v into simple $CTS(v, \lambda_i)$ where, $1 \leq \lambda_i < v-2$, $i = 1, 2, \dots, n$ for odd cases $v \equiv 1, 3, 5 \pmod{6}$. They defined the large set of cyclic triple systems to be a decomposition of all triples of Z_v into indecomposable cyclic triple systems. On the other hand, the near-one-factorization concept was employed to build up a simple 3-fold triple system for the odd cases

(Ibrahim, 2006). Moreover, Matarneh and Ibrahim (2014) utilized the near-two-factorization of a complete multigraph to introduce a CTS $(v, 6)$, called array cyclic design.

In this study, we propose a direct method to construct a new type of CTS $(v, 12)$ with some conditions for $v = 12n+2$ when n is even. This design is called cyclic triple near factorization (briefly CTNF(v)). Then, the construction of CTNF(v) will be used to prove the existence of $[a, b]$ -factorization of $12 K_v$.

MATERIALS AND METHODS

Definitions and preliminaries: Throughout of this study whenever we say λK_v , (or m -cycle), we understand that their vertices are in Z_v with even order and $Z_v^* = Z_v - \{0\}$. In this study, we will obtain the main results by using the difference set method. That already detected to be successful in several cases for the construction of the cyclic (m_1, m_2, \dots, m_r) -cycle system and cyclic triple.

For $\{a, b\} \subset Z_v$ and $a \neq b$, the difference d of a pair $\{a, b\}$ is defined $d = \min\{|a-b|, v-|a-b|\}$, hence, $1 \leq d \leq v/2$. Let, B is a k -subset of Z_v , the differences of B is the multiset $D(B) = \{\min\{|a-b|, v-|a-b|\}, a \neq b \in B\}$. In general, the list of differences of multiset $A = \{B_1, B_2, \dots, B_r\}$ of k -subsets of Z_v is defined as $D(A) = D(B_1) \cup D(B_2) \cup \dots \cup D(B_r)$ such that the union must be understood between multisets (elements must be counted with their respective, multiplicities) (Tian and Wei, 2013).

Definition 1; Abel and Buratti (2006): Let, A be a multiset of k -subsets of Z_v . An A is a (v, k, λ) -difference system if $D(A)$ covers each element of $Z_{v+2/2}^*$ exactly λ times except for the middle difference $(v/2)$ appears $\lambda/2$ times.

Theorem 1; Abel and Buratti (2006): Let, A be a multiset of k -subsets of Z_v . Then A is a starter of cyclic (v, k, λ) -BIBD if and only if A is the (v, k, λ) -difference system.

Definition 2; Tian and Wei (2010): A (v, k, λ) -BIBD (V, B) is indecomposable if there does not exist $\lambda_1 < \lambda$ such that (V, B_1) is (v, k, λ_1) -BIBD and $B_1 \subset B$.

Lemma 1; Colbourn and Rosa (1999): Let, $v \equiv 2 \pmod{4}$, then, the necessary condition for the existence CTS (v, λ) is $\lambda \equiv 0 \pmod{12}$.

Definition 3; Alqadri and Ibrahim (2017): A $(m_1^*, m_2^*, \dots, m_r^*)$ -cycle system of G is a decomposition of G into cycles that have the lengths $\{m_1, m_2, \dots, m_r\}$.

Definition 4; Alqadri and Ibrahim (2017): A set of cycles that generates all cycles of a cyclic $(m_1^*, m_2^*, \dots, m_r^*)$ -cycle system of G by repeated addition of 1 modular v which is called a starter set.

The superscript notation shall be used to describe a starter set of cyclic design. Therefore, $\alpha = \{c_{m_1}^{n_1}, c_{m_2}^{n_2}, c_{m_3}^{n_3}\}$ means that there are n_1 cycles of length m_1 , n_2 cycles of length m_2 , etc. as well as we consider that c_m be the i th m -cycle in starter set δ (Tian and Wei, 2010).

Definition 5; Alqadri and Ibrahim (2017): A differences list from the cycle graph $C_m = (c_0, \dots, c_{m-1})$ is the multiset $D(C_m) = \{\min\{|c_i - c_{i-1}|, v - |c_i - c_{i-1}|\}, i = 1, 2, \dots, m\}$ where, $c_0 = c_m$.

Theorem 2: Let, $\delta = \{C_{m_1}, C_{m_2}, \dots, C_{m_r}\}$ be a set of cycles. A δ is a starter of (m_1, m_2, \dots, m_r) -cycle system λK_v if and only if $D(\delta) = \cup_{i=1}^r D(C_{m_i})$ covers each element of $Z_{v+2/2}^*$ exactly λ times except for the middle difference $(v/2)$ appears $\lambda/2$ times.

RESULTS AND DISCUSSION

Cyclic triple near factorization: In this study, we define a new concepts as a base towards constructing a new types of cyclic 12-fold triple system and $[a, b]$ -factorization.

Definition 6: A cyclic $(m_1^*, m_2^*, \dots, m_r^*)$ -cycle factorization of λK_v is a cyclic $(m_1^*, m_2^*, \dots, m_r^*)$ -cycle system in which its starter (briefly α) satisfies a near-four-factor denoted by $CCF(\lambda K_v, \alpha)$.

Definition 7: A cyclic triple factorization of order v , denoted by CTNF(v) is a $v \times 2(v-1)$ array that satisfies the following conditions:

- The entries in row r form a near-four-factor with focus r
- The triples associated with any row contain no repetitions

The cyclic $(m_1^*, m_2^*, \dots, m_r^*)$ -cycle factorization of $4 K_v$ will be employed to construct CTNF(v). Example 1 explains the construction method of CTNF(v) for $v = 26$.

Example 1: Suppose $G = 4 K_{26}$ and $\alpha = \{C_4^6, C_7^2, C_6^2\}$ is a cycles set of G such that:

Table 1: The list of differences of 4-cycles in $\alpha = \{C_4^6, C_7^2, C_6^2\}$

Cycles	Difference set
(1, 25, 14, 12)	{2, 24, 11, 15, 2, 24, 11, 15}
(2, 24, 15, 11)	{4, 22, 9, 17, 4, 22, 9, 17}
(3, 23, 16, 10)	{6, 20, 7, 19, 6, 20, 7, 19}
(4, 22, 17, 9)	{8, 18, 5, 21, 8, 18, 5, 21}
(5, 21, 18, 8)	{10, 16, 3, 23, 10, 16, 3, 23}
(6, 7, 20, 19)	{1, 25, 13, 13, 1, 25, 13, 13}

Table 2: The list of differences of 7 and 6-cycles in $\alpha = \{C_4^6, C_7^2, C_6^2\}$

Cycles	Difference set
(13, 2, 12, 3, 11, 4, 10)	{11, 15, 10, 16, 9, 17, 8, 18, 7, 19, 6, 20, 3, 23}
(13, 24, 14, 23, 15, 22, 16)	{11, 15, 10, 16, 9, 17, 8, 18, 7, 19, 6, 20, 3, 23}
(6, 1, 5, 17, 19, 18)	{5, 21, 4, 22, 12, 14, 2, 24, 1, 25, 12, 14}
(20, 25, 21, 9, 7, 8)	{5, 21, 4, 22, 12, 14, 2, 24, 1, 25, 12, 14}

Table 3: Construction of CTNF (26)

C_1	C_2	C_3	C_4	C_5	...	C_{26}	C_{20}	C_{31}
0	1, 25	25, 14	14, 12	12, 1	...	9, 7	7, 8	8, 20
1	2, 0	0, 15	15, 13	13, 2	...	10, 8	8, 9	9, 21
2	3, 1	1, 16	16, 14	14, 3	...	11, 9	9, 10	10, 22
...
24	25, 23	23, 12	12, 10	10, 25	...	7, 5	5, 6	6, 18
25	0, 24	24, 13	13, 11	11, 0	...	8, 6	6, 7	7, 19

$$C_{4_1} = (1, 25, 14, 12), C_{4_2} = (2, 24, 15, 11), C_{4_3} = (3, 23, 16, 10)$$

$$C_{4_4} = (4, 22, 17, 9), C_{4_5} = (5, 21, 18, 8), C_{4_6} = (6, 7, 20, 19)$$

$$C_7^* = (13, 2, 12, 3, 11, 4, 10), C_7^{**} = (13, 24, 14, 23, 15, 22, 16)$$

$$C_6^* = (6, 1, 5, 17, 19, 18), C_6^{**} = (20, 25, 21, 9, 7, 8)$$

An easy verification shows that each nonzero integers in Z_{26} appears twice in the cycles of α . Since, a cycle graph is the 2-regular graph, then each vertex in $Z_{26} - \{0\}$ has a 4° . Hence, the cycles of α satisfy a near-four-factor with zero isolated. Furthermore, the differences lists of the cycles in α are listed in Table 1 and 2.

As can be seen from Table 1 and 2, the differences list of α , $D(\alpha)$, covers each integer in Z_{13}^* exactly four times and the middle difference 13 occurs twice. Based on theorem 2, the set of cycles α is a starter of a cyclic $(4^*, 7^*, 6^*)$ -cycle factorization of $4K_{26}$, $CCF(4K_{26}, \alpha)$. Therefore, the construction of $CCF(4K_{26}, \alpha)$ can be viewed as (26×10) array in which $\alpha = \{C_4^6, C_7^2, C_6^2\}$ generates all of its cycles by repeated addition of 1 modular (26).

In order to form CTNF (26), we place the zero element, isolated vertex in the first column then we split the edges of the cycles in each row of $CCF(4K_{26}, \alpha)$ into separated edges by setting each edge in a specific column. Here, we have 26 rows and 50 columns with a column that has an isolated vertex as illustrated in Table 3.

To form triples, we append C_1 with C_i for $2 \leq i \leq 52$. For example, in the first row, we have $\{0, 1, 25\}$, $\{0, 25, 14\}$, $\{0, 14, 12\}$ and by continuing in the same fashion in the remaining rows, we will generate the desired design.

In fact, the construction of $CCF(4K_n, \alpha)$ is the fundamental question of the existence of $CTNF(v)$. To address this question, Alqadri and Ibrahim (2017) provided the starter of $CCF(4K_n, \alpha)$. They constructed the m -cycles for $m > 4$ by connected paths as $C_m = (v_0, P_{2n}, P_{2r})$ such that v_0 is a vertex and $\{P_{2n}, P_{2r}\}$ are a paths of even order. For even paths $\{P_{2n}, P_{2r}\}$ in C_m it can be written as:

$$P_{2n} = [a_1, b_1, a_2, b_2, \dots, a_n, b_n] = [U_{i=1}^n a_i, b_i]$$

$$P_{2r} = [c_1, d_1, c_2, d_2, \dots, c_r, d_r] = [U_{i=1}^r c_i, d_i]$$

where, the edge set of P_{2n} will be represented as:

$$E(P_{2n}) = \begin{cases} \{a_i, b_i\}, & 1 \leq i \leq n, \\ \{a_{i+1}, b_i\}, & 1 \leq i \leq n-1 \end{cases}$$

$$E(P_{2r}) = \begin{cases} \{c_i, d_i\}, & 1 \leq i \leq r, \\ \{c_{i+1}, d_i\}, & 1 \leq i \leq r-1 \end{cases}$$

Thus, the edge set of $C_m = (v_0, P_{2n}, P_{2r})$ will expressed as follows:

$$E(C_m) = E(P_{2n}) \cup E(P_{2r}) \cup \{v_0, a_1\} \cup \{b_n, c_1\} \cup \{v_0, d_r\}$$

The starter set of $CCF(4K_n, \alpha)$ is shown in construction I below.

Construction I: Let, n be an even integer, $n > 2$. Suppose $\alpha = \{C_4^{3n}, C_{4n-1}^2, C_{2n+2}^2\}$ is a cycles set of $4K_{12n+2}$ where the list of 4-cycles is:

$$C_{4_i} = (i, 12n+2-i, 6n+1+i, 6n+1-i), 1 \leq i \leq 3n, i \neq \frac{5n+4}{2}$$

when $i = \frac{5n+4}{2}$, let:

$$C_{4_i} = \left(\frac{5n+4}{2}, 6n+1-\frac{5n+4}{2}, 12n+2-\frac{5n+4}{2}, 6n+1+\frac{5n+4}{2} \right)$$

Whereas $C_{4n-1}^* = (4n+2, P_{4n-2}^*)$ and $C_{4n-1}^{**} = (8n, P_{4n-2}^{**})$ are $(4n-1)$ -cycles in which the paths $\{P_{4n-2}^*, P_{4n-2}^{**}\}$ are represented as follows:

$$P_{4n-2}^* = [6n+1, 2, 6n, 3, \dots, 4n+3, 2n] =$$

$$[U_{i=1}^{2n-1} 6n+2-i, i+1]$$

$$P_{4n-2}^{**} = [6n+1, 12n, 6n+2, 12n-1, \dots, 8n-1, 10n+2] =$$

$$[U_{i=1}^{2n-1} 6n+i, 12n+1-i]$$

Furthermore, $C_{2n+2}^* = (9n, P_2^*, P_{2n-2}^*)$ and $C_{2n+2}^{**} = (3n+2, P_3^{**}, P_{2n-2}^{**})$ are $(2n+2)$ -cycles such that the paths $\{P_3^*, P_{2n-2}^*, P_3^{**}, P_{2n-2}^{**}\}$ are represented as follows:

$$\begin{aligned}
 P_3^* &= [2n+2, 1, 2n+1] \\
 P_{2n-2}^{**} &= [8n+1, 10n-1, 8n+2, 10n-2, \dots, 9n-1, 9n+1] = \\
 &= \left[\bigcup_{i=1}^{n-1} 8n+i, 10n-i \right] \\
 P_3^{**} &= [10n, 12n+1, 10n+1] \\
 P_{2n-2}^* &= [4n+1, 2n+3, 4n, 2n+4, \dots, 3n+3, 3n+1] \\
 &= \left[\bigcup_{i=1}^{n-1} 4n+2-i, 2n+2+i \right]
 \end{aligned}$$

We will use the cycles in construction I to prove the next theorem and also to formulate an algorithm for a starter of 12-fold triple system.

Theorem 3: There exists a cyclic triple factorization of order $12n+2$ for $n \equiv 0 \pmod{2}$.

Proof: We split the proof into two cases as follows:

Case 1: $n = 2$, see example 1.

Case 2: For $n > 2$. There exists CCF $(4K_v, \alpha)$ for $v = 12n+2$. Then, the construction of CCF $(4K_{12n+2}, \alpha)$ can be represented as $(v \times |\alpha|)$ array in which the cycles in each row satisfy the near-four-factor. For producing the CTNF $(12n+2)$, we need to have $12n+2$ rows and $2(12n+1)$ columns and a column with an isolated vertex. In the construction of CCF $(4K_{12n+2}, \alpha)$, we place the isolated vertex in the first column and then partition the edge set of the cycles in each row into separated edges by setting every edge in a column. On the other hand, the number of columns equal to the edges cycles of $\alpha = \{C_4^{3n}, C_{4n-1}^2, C_{2n+2}^2\}$ which is calculated as follows:

$$4 \times (3n) + 2 \times (4n-1) + 2 \times (2n+2) = 2(12n+1)$$

In order to form triples, appending the first column with each other columns. Since, there are no identical edges in each row, then all the associated triples in each row will be distinct. The triples in the first row in the construction of CTNF $(12n+2)$ is considered a starter triples (briefly A).

Theorem 4: For $v = 12n+2$ where, $n \equiv 0 \pmod{2}$, there exists a cyclic 12-fold triple system has cyclic triple near factorization of order v .

Proof: In order to prove this theorem, we need to calculate the differences list of the starter triples A of

CTNF $(12n+2)$. Depending on theorem 3, all triples in A have formed as $\{0, e_i\}, 1 \leq i \leq 2(12n+1)$ where $e_i = \{c_{1,i}, c_{2,i}\}$ is the edge set of α . Then, the differences list of A is calculated as follows:

$$D(A) = \bigcup \left\{ \begin{aligned} &d(e_i) \cup d(0, c_{j,i}), 1 \leq i \leq \\ &2(12n+1) \text{ and } 1 \leq j \leq 2 \end{aligned} \right\} \quad (1)$$

Indeed, the cycles of α in the construction I is the starter of cyclic $(4^*, (4n-1)^*, (2n+2)^*)$ -cycle system of $4K_{12n+2}$ (Alqadri and Ibrahim, 2017). Based on theorem 2 $D(\alpha) = \bigcup_{i=1}^{2(12n+1)} d(e_i)$ covers each nonzero element of Z_{6n+1} four times and the middle difference $\{6n+1\}$ occurs twice.

Furthermore, α satisfies the near-four-factor of $4K_{12n+2}$ with isolated zero integer, i.e., $\{1, 2, \dots, v/2, (v/2+1), (v/2+2), \dots, 12n+1\}$ appear four times as end points of the edge set of α . Hence, for any end point $c_{j,i}$ of an edge $e_i, 1 \leq i \leq 2(12n+1)$ and $1 \leq j \leq 2$, we find out:

$$d\{0, c_{j,i}\} = \min\{c_{j,i}, v - c_{j,i}\} = \begin{cases} c_{j,i} & c_{j,i} \geq \frac{v}{2} \\ v - c_{j,i} & c_{j,i} < \frac{v}{2} \end{cases}$$

Then, every integer of $\{1, 2, \dots, v/2, (v/2-1), (v/2-2), \dots, 2, 1\}$ occur four times in $\bigcup_{i,j} d(0, c_{j,i})$ and this means $\bigcup_{i,j} d(0, c_{j,i})$ contained all integers of Z_{6n+1} 8 times and the integer $\{6n+1\}$ four times. From Eq. 1, we conclude that $D(A)$ covers each nonzero element of Z_{6n+1} 12 times and the middle difference $\{6n+1\}$ occurs 6 times. Therefore, A is a starter of cyclic 12-fold triple system based on theorem 1.

Lemma 2: A CTNF(v) is indecomposable $(v, 3, 12)$ -BIBD for $v = 12n+2$.

Proof: Based on lemma 1, the smallest λ for the existence of λ -fold triple system of order $v = 12n+2$ is 12. Then, there is no $\lambda < 12$ such that CTS(v, λ) exists.

Algorithm of starter triples of CTNF(12n+2): In this study, we use the starter cycles of CCF $(4K_{12n+2}, \alpha)$ to construct and formulate the algorithm of starter triples A of CTNF $(12n+2)$. The process of formulating an algorithm for the starter triples A will be split into two cases depending on $n = 2$ or $n > 2$.

Case 1: For $n = 2$. By virtue example 1, it follows that a starter triples of CTNF (26) is:

$$A = A_1 \cup A_2$$

Such that:

$$A_1 = \begin{cases} \{0, i, 26-i\} \cup \{0, 26-i, 13+i\} & 1 \leq i \leq 5 \\ \{0, 13+i, 13-i\} \cup \{0, 13-i, i\} & 1 \leq i \leq 5 \\ \{0, 14-i, i+1\} \cup \{0, 13-i, i+1\} & 1 \leq i \leq 3 \\ \{0, 12+i, 25-i\} \cup \{0, 13+i, 25-i\} & 1 \leq i \leq 3 \end{cases}$$

$$A_2 = \left\{ \{0, 6, 7\}, \{0, 7, 20\}, \{0, 20, 19\}, \right. \\ \left. \{0, 19, 6\}, \{0, 13, 10\}, \{0, 13, 16\} \right\}$$

Case 2: $n \equiv 0 \pmod{2}$ where, $n \geq 4$. The generated triples by appending zero integer isolated vertex, to each edge in the edge set of the cycles in the construction I are formed as subsets. We begin with the 4-cycles (C_4^{3n}) as follows:

$$S_1 = \left\{ \left\{ 0, \frac{5n+4}{2}, \frac{7n-2}{2} \right\}, \left\{ 0, \frac{7n+2}{2}, \frac{19n}{2} \right\}, \right. \\ \left. \left\{ 0, \frac{19n}{2}, \frac{17n-6}{2} \right\}, \left\{ 0, \frac{17n+6}{2}, \frac{5n-4}{2} \right\} \right\}$$

$$S_2 = \left\{ \{0, i, 12n+2-i\} \quad 1 \leq i \leq 3n, i \neq \frac{5n+4}{2} \right\}$$

$$S_3 = \left\{ \{0, 12n+2-i, 6n+1+i\} \quad 1 \leq i \leq 3n, i \neq \frac{5n+4}{2} \right\}$$

$$S_4 = \left\{ \{0, 6n+1-i, 6n+1+i\} \quad 1 \leq i \leq 3n, i \neq \frac{5n+4}{2} \right\}$$

$$S_5 = \left\{ \{0, i, 6n+1-i\} \quad 1 \leq i \leq 3n, i \neq \frac{5n+4}{2} \right\}$$

According to, the edge set of $(4n-1)$ -cycles, the list of generated triples from $\{C_{4n-1}^*, C_{4n-1}^{**}\}$ will be as follows:

$$S_6 = \left\{ \{0, i+1, 6n+2-i\} \quad 1 \leq i \leq 2n-1 \right\}$$

$$S_7 = \left\{ \{0, i+1, 6n+1-i\} \quad 1 \leq i \leq 2n-2 \right\}$$

$$S_8 = \left\{ \{0, 12n+1-i, 6n+1+i\} \quad 1 \leq i \leq 2n-2 \right\}$$

$$S_9 = \left\{ \{0, 12n+1-i, 6n+i\} \quad 1 \leq i \leq 2n-1 \right\}$$

$$S_{10} = \left\{ \{0, 4n+2, 2n\}, \{0, 4n+2, 6n+1\}, \right. \\ \left. \{0, 10n+2, 8n\}, \{0, 8n+6n+1\} \right\}$$

Meanwhile, the following subsets represent the produced triples from the cycles $\{C_{2n+2}^*, C_{2n+2}^{**}\}$:

$$S_{11} = \left\{ \{0, 10n-i, 8n+i\} \quad 1 \leq i \leq n-1 \right\}$$

$$S_{12} = \left\{ \{0, 10n-i, 8n+1+i\} \quad 1 \leq i \leq n-2 \right\}$$

$$S_{13} = \left\{ \{0, 2n+2+i, 4n+2-i\} \quad 1 \leq i \leq n-1 \right\}$$

$$S_{14} = \left\{ \{0, 2n+2+i, 4n+1-i\} \quad 1 \leq i \leq n-2 \right\}$$

$$S_{15} = \left\{ \{0, 9n+1, 9n\}, \{0, 9n, 2n+2\}, \{0, 2n+2, 1\}, \{0, 1, 2n+1\}, \right. \\ \left. \{0, 2n+1, 8n+1\}, \{0, 3n, 3n+2\}, \{0, 3n+2, 10n\}, \right. \\ \left. \{0, 10n, 12n+1\}, \{0, 12n+1, 10+1\}, \{0, 10n+1, 4n+1\} \right\}$$

For simplicity, we will combine the subsets together which have a relationship between their triples. As a result, the algorithm of the starter triples A of CTNF ($12n+2$) can be formulated as:

$$A = A_1 \cup A_2$$

Such that A_1 and A_2 are computed below:

$$A_1 = \begin{cases} \{0, i, 12n+2-i\} & 1 \leq i \leq 6n, \text{ if } i \notin \left\{ \frac{5n+4}{2}, \frac{7n-2}{2} \right\} \\ \{0, 12n+2-i, 6n+1+i\} \cup \{0, 6n+1-i, i\} & 1 \leq i \leq 3n, \text{ if } i \notin \left\{ \frac{5n+4}{2} \right\} \\ \{0, 6n+i, 12n+1-i\} & 1 \leq i \leq 3n-1, \text{ if } i \notin \{2n, 2n+1\} \\ \{0, 6n+2-i, i+1\} & 1 \leq i \leq 3n-1, \text{ if } i \notin \{2n, 2n+1\} \\ \{0, 6n+1-i, i+1\} \cup \{0, 6n+1+i, 12n+1-i\} & 1 \leq i \leq 2n-2 \\ \{0, 8n+i, 10n-i\} \cup \{0, 4n+2-i, 2n+2+i\} & 1 \leq i \leq 2n-1 \end{cases}$$

$$A_2 = \left\{ \left\{ 0, \frac{5n+4}{2}, \frac{7n-2}{2} \right\}, \left\{ 0, \frac{7n-2}{2}, \frac{19n}{2} \right\}, \left\{ 0, \frac{19n}{2}, \frac{17n+6}{2} \right\}, \left\{ 0, \frac{17n+6}{2}, \frac{5n+4}{2} \right\}, \{0, 4n+2, 2n\} \right. \\ \{0, 4n+2, 6n+1\}, \{0, 8n, 6n+1\}, \{0, 8n+10n+2\}, \{0, 9n+1, 9n\}, \{0, 9n, 2n+2\}, \{0, 2n+2, 1\}, \\ \{0, 1, 2n+2\}, \{0, 2n+1, 8n+1\}, \{0, 3n, 3n+2\}, \{0, 3n+2, 10n\}, \{0, 10n, 12n+1\}, \\ \left. \{0, 12n+1, 10n+1\}, \{0, 10n+1, 4n+1\} \right\}$$

Theorem 5: There exist $[8, 4(v-1)]$ -factorization of $12K_v$ for $v = 12n+2, n \equiv 0 \pmod{2}$.

Proof: From the construction of $CTNF(12n+2)$, it can be viewed the triples in $CTNF(12n+2)$ as $v \times 2(v-1)$ array in which the vertex r is included in $2(v-1)$ distinct triples and each another vertex in Z_{12n+2} occurs in exactly four triples in row r . As pointed out in the introduction, the $(v, 3, \lambda)$ -BIBD can be regarded as a decomposition of λK_v into copies of K_3 . Since, any vertex in K_3 has a degree two, then any vertex has a degree either 8 or $4(v-2)$ in every row r for $0 \leq r \leq v-1$. So, we can consider each row r_i in $CTNF(12n+2)$ is $[8, 4(v-1)]$ -factor of $12K_v$ where, the element v_i has a degree 4 $(v-1)$ and another vertex has a 8° .

CONCLUSION

In this study, we have defined a new method to construct a new cyclic 12-fold triple system. We employed a cyclic $(m_1^*, m_2^*, \dots, m_r^*)$ -cycle system and near-four-factorization of $4K_v$ to construct cyclic triple factorization, $CTNF(v)$. We have also, proven the existence of $CTNF(v)$ for $v = 12+2$ when n is even besides an algorithm for its starter triples. Therefore, we used the related results of $CTNF(v)$ to demonstrate the existence of $[8, 2(v-1)]$ -factorization of $12K_v$. We expect the construction of $CTNF(v)$ is a simple design and can be extended for $v = 12+2$ when n is odd and also for $v \equiv 6, 10 \pmod{12}$.

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