# Cyclic Triple Factorization

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# Cyclic Triple Factorization

Mowafaq Alqadri, Haslinda Ibrahim, Sharmila Karim

*Abstract*—**This article aims to present a novel method, namely wheel partition technique, for constructing a new cyclic 12-fold triple system called cyclic triple factorization denoted by $CTF(v)$. We prove the existence of $CTF(v)$ for $v = 12n + 10$. Then, an arrangement of $v \times 2(v-1)$ triples of $CTF(v)$ is developed using the idea of decomposition of wheel graph into triangles (triples). Moreover, the starter triples algorithms of $CTF(v)$ are formulated to generate all triples.**

*Index Terms*—**complete multigraph, near four factorization, triple system, wheel graph.**

## I. INTRODUCTION

Throughout this paper, all graphs are considered undirected with vertices in a cyclic group $Z_v$. The standard notations of graph theory are used so that $\lambda K_v$, $C_m$ and $W_n$, respectively, denote the complete multigraph on $v$ vertices, the $m$-cycle and the wheel graph of order $n$. As usual speaking of the wheel graph $W_n = c_0 + (c_1, c_2, \dots, c_{n-1})$, means that contains a cycle of order $n-1$, and each vertex in the cycle is adjacent to another new vertex, $c_0$, which is known as hub.

A $(\lambda K_v, \mathcal{Y})$-design is a decomposition of $\lambda K_v$ into a multiset of subgraphs $\mathcal{Y} = \{H_1, \dots, H_r\}$. Automorphism group of $(\lambda K_v, \mathcal{Y})$-design $\Pi$ is a group of bijections on $V(\lambda K_v) = Z_v$ fixing $\mathcal{Y}$. If there is an automorphism $\alpha \in \Pi$ that is a permutation of order $v$, it is called cyclic. Thus, the automorphism can be expressed by

$$\alpha: i \longrightarrow i+1 \ (mod\ v) \text{ or } \alpha: (0, 1, \dots, v-1).$$

A starter of $(\lambda K_v, \mathcal{Y})$-design is a multiset of $\mathcal{Y}$ that generates all the graphs of $\mathcal{Y}$ by repeated addition of 1 modulo $v$ [1]. In particular, the $(K_v, \mathcal{Y})$-design is called an $m$-cycle system of order $v$ if $\mathcal{Y}$ is a collection of $m$-cycles. The existence question of $m$-cycle system of order $v$ has been solved in [2]-[3]. Recently, Bryant et al. [4] showed the necessary and sufficient conditions for the decomposition of $K_v$ into cycles of various orders, or into cycles of distinct orders and a perfect matching. More recently, the necessary and sufficient conditions have been extended to decompose $\lambda K_v$ into cycles of varying lengths [5].

A $k$-factor of a graph $G$ is a spanning subgraph whose vertices have a degree $k$. While a near-$k$-factor is a spanning subgraph in which all vertices have a degree $k$ with exception of one vertex (isolated vertex) which has a degree zero. The partition of edge set of a graph $G$ into $k$-factor (respectively, near- $k$-factor) is called a $k$- factorization (respectively, near- $k$- factorization). More general results on near-$\lambda$-factorization of $\lambda K_v$ have been presented in [6]-[7].

A balanced incomplete block design is a pair $(V, \mathcal{B})$ where $V$ is a finite set of $v$ points and $\mathcal{B}$ is a multiset of $k$-subsets of $V$ called blocks such that each 2-subset of $V$ is contained in precisely $\lambda$ blocks. Such design is denoted $(v, k, \lambda)$-$BIBD$. A $\lambda$-fold triple system is $(v, 3, \lambda)$-$BIBD$ and denoted $TS(v, \lambda)$. On other words, we can say that a $\lambda$-fold triple system is a decomposition for $\lambda K_v$ into edge disjoint triangles. The pair $(V, \mathcal{B})$ is called a cyclic triple system, $CTS(v, \lambda)$, if $V = Z_v$ and if $B = \{c_1, c_2, c_3\} \in \mathcal{B}$ then $B + 1 = \{c_1 + 1, c_2 + 1, c_3 + 1\}$ is also in $\mathcal{B}$. The orbit of the triple $B$, denoted by $orb(B)$, is the set of all distinct triples in the collection $\{B + i \mid i \in Z_v\}$. The length of orbit $B$ is its cardinality i.e., $orb(B) = k$, where $k$ is the minimum positive integer such that $B + k = B$. If the orbit of $B$ is $v$, it is called a strictly; otherwise it is short. When $v \not\equiv 0\ (mod\ 3)$ then there is no short orbit of block [8].

The existence of a $\lambda$-fold triple system of order $v$ for any possible parameters $\lambda$ and $v$ is considered an interesting problem due to its nice combinatorial properties and its relationship to optical orthogonal codes [9]. For more, readers can refer to [10]-[12]. In [13], Colbourn and Colbourn studied the existence of cyclic triple system over $Z_v$ when $v \equiv 1, 3\ (mod\ 6)$. While the necessary conditions for the existence $CTS(v, \lambda)$ have been given by Colbourn and Rosa [14].

One of the latest triple systems is triad design, which is concerned in arranging all triples of $Z_v$ according to some constraints. Ibrahim and Wallis employed near-one-factorization to present a new type of triple system that is called compatible factorization which is used in building up the triad design. They proved the existence of the triad design of $Z_v$ for $v \equiv 1, 5(mod\ 6)$ [15]. Moreover, the algorithms of starter of triad design have been formulated for the cases $v \equiv 1, 5(mod\ 6)$ [16]-[17].

On the other hand, a new method for decomposing all triples of $Z_v$ into cyclic triple systems for the case of odd $v$, $v \equiv 1, 3, 5(mod\ 6)$, has been introduced. The large set of cyclic triple systems has been defined to be a decomposition of all triples of $Z_v$ into indecomposable cyclic systems [8]-[9].

In this paper, a new method is presented to construct a new type of cyclic $\lambda$-fold triple system of order $12n + 10$, called cyclic triple factorization. This method depends on employing a cyclic $(\lambda K_v, \mathcal{Y})$-design when $\lambda = 4$ and $\mathcal{Y}$ is a collection of cycles of varying lengths satisfying near-four-

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factorization.

## II. PRELIMINARIES

This section recalls briefly some definitions, notations and preliminary results that we used in the sequel. In this paper, we consider $Z_v$ with even order and $Z_v^* = Z_v - \{0\}$. For $a \neq b \in Z_v$, the difference $d$ of a pair $\{a, b\}$ is defined as $d = \{min\{|a-b|, v-|a-b|\}\}$. So, the difference of any pair of points in $Z_v$ is not exceeding $\left(\frac{v}{2}\right)$, $\left(1 \leq d \leq \frac{v}{2}\right)$. Let $B$ be a $k$-subset of $Z_v$, the list of differences from $B$ is the multiset $D(B) = \{min\{|a-b|, v-|a-b|\}, a \neq b \in B\}$. Generally, the list of differences of multiset $\mathcal{A} = \{B_1, B_2, \dots, B_r\}$ of $k$-subsets of $Z_v$ is defined as $D(\mathcal{A}) = \bigcup_{i=1}^{r} D(B_i)$. The following concepts have been presented in [18].

**Definition 1.** Let $\mathcal{A}$ be a multiset of $k$-subsets of $Z_v$. An $\mathcal{A}$ is a $(v, k, \lambda)$-difference family if $D(\mathcal{A})$ covers each element of $Z^*_{\frac{v+2}{2}}$ exactly $\lambda$ times except for the middle difference $\left(\frac{v}{2}\right)$ appears $\frac{\lambda}{2}$ times.

**Theorem 1 .** Let $\mathcal{A}$ be a multiset of $k$-subsets of $Z_v$. Then $\mathcal{A}$ is a starter of cyclic $(v, k, \lambda)$-$BIBD$ if and only if $\mathcal{A}$ is a $(v, k, \lambda)$-difference family.

**Theorem 2 .** The existence of cyclic $(v, k, \lambda)$-$BIBD$ under $Z_v$ is completely equivalent to the existence of a $(v, k, \lambda)$-difference family in $Z_v$.

On the other hand, Let $H$ be a subgraph of a graph $G$ of order $v$ and let $N_H(x)$ be a multiset of neighbours of $x$ in $H$. Then the list of differences of $H$ is $D(H) = \{min\{|x-y|, v-|x-y|\}, x \in V(H), y \in N_H(x)\}$. More generally, given a set $\delta = \{H_1, H_2, \dots, H_r\}$ of subgraphs of $G$, the list of differences from $\delta$ is defined by $D(\delta) = \bigcup_{i=1}^{r} D(H_i)$.

**Definition 2 .** Let $\delta$ be a multiset of subgraphs $\lambda K_v$. A $\delta$ is a $(G, \mathcal{Y})$- difference family if $D(\delta)$ covers each element of $Z^*_{\frac{v+2}{2}}$ exactly $\lambda$ times except for the middle difference $\left(\frac{v}{2}\right)$ appears $\frac{\lambda}{2}$ times.

As a particular result of the theory developed in [18], we have **Theorem 3 .** Let $\delta$ be a multiset of subgraphs $\lambda K_v$. Then $\delta$ is a starter of cyclic $(G, \mathcal{Y})$- design if and only if $\delta$ is a $(G, \mathcal{Y})$-difference family.

## III. INTRODUCTORY RESULTS

In this section, we introduce some definitions and results required to establish our main aims in the next sections.

**Definition 3.** A $(m_1^*, m_2^*, \dots, m_r^*)$-cycle system of $G$ is a $(G, \mathcal{Y})$- design in which $\mathcal{Y}$ is a collection of cycles of length $\{m_1, m_2, \dots, m_r\}$.

**Definition 4.** A cyclic $(m_1^*, m_2^*, \dots, m_r^*)$-cycle factorization of $\lambda K_v$ is a $(m_1^*, m_2^*, \dots, m_r^*)$-cycle system in which the starter (briefly $\delta$) is a near-$\lambda$-factor denoted by $CCF(\lambda K_v, \delta)$.

Following Tian and Wei [9], we use the superscript notation to describe a starter set of cyclic design. Therefore, $\delta = \{C_{m_1}^{n_1}, C_{m_2}^{n_2}, \dots, C_{m_r}^{n_r}\}$ means that there are $n_1$ cycles of length $m_1$, $n_2$ cycles of length $m_2$, etc., as well as we consider that $C_{m_i}$ be the $i$-th $m$-cycle in starter set $\delta$.

**Lemma 1.** Let $G$ be graph of order $v$. Let $n > 0$ be an even integer and $\mathcal{C}$ be a set of cycles of $G$. Then $\mathcal{C}$ is a near-$n$-factor of $G$ if and only if the vertex set of $\mathcal{C}$ covers every element of $G$ exactly $\frac{n}{2}$ times except one vertex.

**Proof.** Let $\mathcal{C} = \{C_{m_1}, C_{m_2}, \dots, C_{m_r}\}$ be a set of cycles that satisfies a near-$n$-factor then each vertex of $\mathcal{C}$ has a degree $n$ except the isolated vertex. Let $x \in V(G)$ and $x$ is not isolated vertex in $\mathcal{C}$. Then, the degree of $x$ in $\mathcal{C}$ is

$$deg_{\mathcal{C}}(x) = \sum_{i=1}^{r} deg_{C_{m_i}}(x)$$

where $deg_{\mathcal{C}}(x)$ and $deg_{C_i}(x)$ denote the degree of $x$ in $\mathcal{C}$ and $C_{m_i}$ respectively. Since a cycle graph is a 2-regular graph, then $deg_{C_{m_i}}(x) = 2$ or 0 according to whether or not $x$ is a vertex of $C_{m_i}$, $1 \leq \mathrm{i} \leq r$. Suppose the number of cycles in $\mathcal{C}$ that contains $x$ is $k$. Then, we have:

$$\begin{aligned} deg_{C_{m_i}}(x) &= 2 + 2 + \dots + 2, \\ &= 2 \times k. \end{aligned}$$

Since $deg_{\mathcal{C}}(x) = n$, then $k = \frac{n}{2}$.

The next task is to show that if each vertex of $G$ appears $\frac{n}{2}$ times except one vertex in $\mathcal{C} = \{C_{m_1}, C_{m_2}, \dots, C_{m_r}\}$ then the cycles of $\mathcal{C}$ satisfy near-$n$-factor. Consider $y$ is a vertex of $G$ that does not appear in $\mathcal{C}$ and $x \in V(G), x \neq y$. So, $x$ occurs $\frac{n}{2}$ times in the cycles of $\mathcal{C}$. Hence, the degree of $x$ in $\mathcal{C}$ is calculated as

$$deg_{\mathcal{C}}(x) = \sum_{i=1}^{r} deg_{C_{m_i}}(x)$$

such that

$$deg_{C_{m_i}}(x) = \begin{cases} 2, & x \in C_{m_i}, \\ 0, & x \notin C_{m_i}. \end{cases}$$

Since $x$ appear $\frac{n}{2}$ times in the cycles of $\mathcal{C}$ then

$$deg_{\mathcal{C}}(x) = \underbrace{2 + 2 + \dots + 2}_{\frac{n}{2}\ times} = 2 \times \frac{n}{2} = n.$$

Therefore, $\mathcal{C} = \{C_{m_1}, C_{m_2}, \dots, C_{m_r}\}$ forms a near-$n$-factor with isolated $y$. □

**Remark 1.** The set of cycles cannot fulfill a near-$n$-factor when $n$ is odd since the cycle is a 2-regular graph. Thus, the degree of any vertex in the cycle will be even.

Consistent with the definition of wheel graph, the edge set of wheel graph of order $n$, $W_n = c_0 + (c_1, \dots, c_{n-1})$, is divided into two sets as follows:

$$E(W_n) = E\left(K_{(1,n-1)}\right) \cup E(C_{n-1}).$$

Where

$E\left(K_{(1,n-1)}\right) = \{c_0c_i \mid 1 \le i \le n-1\}$;

$E(C_{n-1}) = \{c_ic_{i+1} \mid 1 \le i \le n-1\}$ where $c_n = c_1$.

**Definition 5.** Let $W_n$ be a wheel of a graph $G$ of order $v$. The list of differences of $W_n$, denoted by $D(W_n)$, is the multiset $D(W_n) = D(C_{n-1}) \cup D\left(K_{(1,n-1)}\right)$ such that

$D(C_{n-1}) =$
$\{min\{|c_i - c_{i-1}|, v - |c_i - c_{i-1}|\} \mid 1 \le i \le n\}, c_n = c_0$;

$D\left(K_{(1,n-1)}\right) =$
$\{min\{|c_i - c_0|, v - |c_i - c_0|\} \mid 1 \le i \le n-1\}$.

We call $D(C_{n-1})$ and $D\left(K_{(1,n-1)}\right)$ the cycle differences $(CD)$ and internal differences $(ID)$, respectively.

**Lemma 2**. Let $v$ be an even integer and $\mathcal{W}$ be a set of wheels of a graph of order $v$. If the associated cycles with wheels in $\mathcal{W}$ form a near-four-factor, then the internal differences, $(ID)$, of $\mathcal{W}$ covers each element in $Z^*_{\frac{v+2}{2}}$ four times except the middle difference $\frac{v}{2}$ twice.

**Proof.** Let $\mathcal{W} = \left\{c_0 + C_{m_1}, c_0 + C_{m_2}, \dots, c_0 + C_{m_r}\right\}$ be a set of wheels of graph of order $v$ such that the set of cycles $\left\{C_{m_i}, 1 \le i \le r\right\}$ form a near-four-factor with isolated $c_0$. Then, the internal differences of $\mathcal{W}$, $(ID)$, is determined as follows:

$$D\left(K_{(1,m_i)}\right) = \{min\{|c_j - c_0|, v - |c_j - c_0|\} \mid c_j \in C_{m_i}, 1 \le i \le r, 1 \le j \le m_i\}$$

$$D\left(K_{(1,m_i)}\right) = \begin{cases} |c_j - c_0|, |c_j - c_0| \le \frac{v}{2}, \ c_j \in C_{m_i}, 1 \le i \le r, 1 \le j \le m_i; \\ v - |c_j - c_0|, |c_j - c_0| > \frac{v}{2}, c_j \in C_{m_i}, 1 \le i \le r, 1 \le j \le m_i. \end{cases}$$

Since the cycles $\left\{C_{m_i}, 1 \le i \le r\right\}$ form a near-4-factor, then the vertex set of cycles $\left\{C_{m_i}, 1 \le i \le r\right\}$ covers each element of $Z_v$ twice except $c_0$ based on Lemma 1.

Now if we label $c_0$ by "0", then every vertex of the following set:

$$\left\{1, 2, \dots, \left(\frac{v}{2} - 1\right), \frac{v}{2}, \left(\frac{v}{2} + 1\right), \dots, (v-2), (v-1)\right\}$$

will appear as $c_j \in C_{m_i}$ twice. Therefore, $(ID)$ can be written as:

$$D\left(K_{(1,m_i)}\right) = \begin{cases} c_j, & c_j \le \frac{v}{2}, \ c_j \in C_{m_i}, 1 \le i \le r; \\ v - c_j, & c_j > \frac{v}{2}, \ c_j \in C_{m_i}, 1 \le i \le r. \end{cases}$$

Thus, every element in the multiset of

$$\left\{1, 2, \dots, \left(\frac{v}{2} - 1\right), \frac{v}{2}, \left(\frac{v}{2} - 1\right), \dots, 2, 1\right\}$$

will be shown twice. Then $D\left(K_{(1,m_i)}\right)$ covers all the nonzero elements of $Z^*_{\frac{v+2}{2}}$ four times except the middle difference $\frac{v}{2}$ occur twice. □

## IV. CYCLIC TRIPLE FACTORIZATION

In this section we propose a new type of triple system, namely cyclic triple factorization, which contributes to arrange $v \times 2(v-1)$ triples into $v$ rows.

**Definition 6.** A cyclic triple factorization of order $v$, denoted by $CTF(v)$, is a way of arranging $v \times 2(v-1)$ triples into $v$ rows such that it satisfies the following conditions:

(i) Object $r$ appears precisely $2(v-1)$ times in each row $r$.
(ii) Each object except $r$ appears four times in each row $r$.
(iii) The triples associated with row $r$ contains no repetitions.

In order to construct the cyclic triple factorization, the starter of cyclic $(m_1^*, \dots, m_r^*)$-cycle factorization of $4K_v$ is employed. Let us provide an example to illustrate the construction method of $CTF(v)$ by exploiting cycles set.

**Example 1.** Let $G = 4K_{22}$ and $\delta = \{C_4^5, C_{11}^2\}$ is a set of cycles of $G$ such that:

$C_{4_1} = (1, 21, 12, 10)$; $C_{4_2} = (2, 20, 13, 9)$; $C_{4_3} = (3, 19, 14, 8)$; $C_{4_4} = (4, 18, 7, 15)$; $C_{4_5} = (5, 17, 16, 6)$;

$C_{11_1} = (21, 2, 11, 3, 10, 4, 9, 6, 8, 7, 17)$;

$C_{11_2} = (1, 20, 11, 19, 12, 18, 13, 16, 14, 15, 5)$.

Easily, it can be observed that the differences list of $\delta$ covers each nonzero element of $Z_{12}$ four times except the middle difference 11 which appears twice. Thus, $\delta$ is considered a starter of cyclic $(4^*, 11^*)$-cycle system based on Theorem 3. Furthermore, it could be noticed that each nonzero element in $Z_{22}$ occurs twice in the cycles of $\delta$. From Lemma 1, the cycles of $\delta$ form a near-four-factor with focus zero element. Consequently, the cyclic $(4^*, 11^*)$-cycle factorization of $4K_{22}$, $CCF(4K_{22}, \delta)$, is $(22 \times 7)$ array in which $\delta = \{C_4^5, C_{11}^2\}$ generates all of its cycles by repeated addition of 1 modulo (22).

To construct $CTF(22)$ using the construction of $CCF(4K_{22}, \delta)$, we set the isolated vertex in the first column, then we partition the edges of the cycles in each row of $CCF(4K_{22}, \delta)$ into separated edges by placing each edge in a specific column. Here we have 22 rows and 42 columns (number of edges set of $\delta$) with a column that has an isolated vertex as shown in Table I.

TABLE I
PARTITION OF EDGE SET OF THE CYCLES IN $CCF(4K_{22}, \delta)$ INTO SEPARATED EDGES

| $Col_1$ | $Col_2$ | $Col_3$ | ⋯ | $Col_{41}$ | $Col_{42}$ | $Col_{43}$ |
|---|---|---|---|---|---|---|
| 0 | 1, 21 | 21, 12 | ⋯ | 14, 15 | 15, 5 | 5, 1 |
| 1 | 2, 0 | 0, 13 | ⋯ | 15, 16 | 16, 6 | 6, 2 |
| 2 | 3, 1 | 1, 14 | ⋯ | 16, 17 | 17, 7 | 7, 3 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 20 | 21, 19 | 19, 9 | ⋯ | 12, 13 | 13, 3 | 3, 21 |
| 21 | 0, 20 | 20, 10 | ⋯ | 13, 14 | 14, 4 | 4, 0 |

To construct $CTF(22)$, append the isolated vertex $r$ to the endpoints of each edge in row $r$ for $0 \le r \le 21$. Since the cycles in each row of the construction of $CCF(4K_{22},\delta)$ form a near-four-factor, then every vertex has a degree four except the isolated vertex. Thus, every vertex will appear four times except the isolated vertex $r$ in each row $r$ for $0 \leq r \leq 21$. Moreover, all triples in each row are distinct because there is no identical edges in each row $r$ for $0 \leq r \leq 21$ as shown in Table II.

TABLE II
$CTF(22)$

| $Col_1$ | $Col_2$ | $\cdots$ | $Col_{41}$ | $Col_{41}$ | $Col_{42}$ |
|---|---|---|---|---|---|
| $\{0,1,21\}$ | $\{0,21,12\}$ | $\cdots$ | $\{0,14,15\}$ | $\{0,15,5\}$ | $\{0,5,1\}$ |
| $\{1,2,0\}$ | $\{1,0,13\}$ | $\cdots$ | $\{0,15,16\}$ | $\{1,16,6\}$ | $\{1,6,2\}$ |
| $\vdots$ | $\vdots$ | $\cdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\{21,0,20\}$ | $\{21,20,11\}$ | $\cdots$ | $\{0,13,14\}$ | $\{21,14,4\}$ | $\{21,4,0\}$ |

The construction of starter $\delta$ of cyclic $(4^*,(6n+5)^*)$-cycle factorization of $4K_{12n+10}$ , $CCF(4K_{12n+10},\delta)$, has been introduced in [19] as shown in Fig. 1 and 2, in which the cycles of order $6n+5$ were formulated as connected paths. The starter of $CCF(4K_{12n+10},\delta)$ will be used mainly to construct a starter of cyclic triple factorization of order $12n+10$.

In the following, we prove the existence of $CTF(v)$ for the general case when $v = 12n+10$.

**Theorem 4.** Let $n$ be an integer. There exists a cyclic triple factorization of order $12n+10$.

**Proof.** Let $\delta = \{C_4^{3n+2}, C_{6n+5}^2\}$ be the set of cycles of $4K_{12n+10}$ as shown in Fig. 1 and 2 then $\delta$ is a starter of $CCF(4K_{12n+10},\delta)$ [19]. The construction of $CCF(4K_{12n+10},\delta)$ is $\left((12n+10)\times|\delta|\right)$ an array such that is generated by $\delta = \{C_4^{3n+2}, C_{6n+5}^2\}$ with a property that the cycles in each row $r$ form a near-four-factor with isolated $r$ for $0 \leq r < 12n+10$.

To construct $CTF(12n+10)$, we need to have $12n+10$ rows and $2\,(12n+9)$ columns based on Definition 6. In $CCF(4K_{12n+10},\delta)$ construction, we partition the edge set of the cycles in each row into separated edges by setting every edge in a column. Thus, the number of columns in $CTF(12n+10)$ is equal to the number of edges in $\delta = \{C_4^{3n+2}, C_{6n+5}^2\}$. Since $\delta = \{C_4^{3n+2}, C_{6n+5}^2\}$ has $(3n+2)$ cycles of order four and two cycles of order $(6n+5)$, then the number of edges in $\delta = \{C_4^{3n+2}, C_{6n+5}^2\}$ is calculated by the following equation:

$$4\times(3+2)+2\times(6n+5)=2(12n+9) \tag{1}$$

In order to form triples, append the isolated vertex $r$ with the endpoints of every edge in row $r$ for $0 \leq r \leq 10n+9$. Then, from Equation (1), we can see that the isolated vertex will appear in $2(12n+9)$ triples and other vertices will appear four times since the cycles of each row form a near-four-factor with isolated $r$. Since there no edges have the same endpoints in each row $r$ of $CCF(4K_{12n+10},\delta)$, then all the associated triples in each row $r$ of $CTF(12n+10)$ will be distinct. □

Now, it is natural to ask if the construction of cyclic triple factorization forms the cyclic $\lambda$-fold triple system, $CTS(v,\lambda)$. In order to prove that $CTF(v)$ is $CTS(v,\lambda)$, we must show that $CTF(v)$ has a balanced property, namely every pair of distinct elements of $v$ belongs to exactly $\lambda$ triples. In this way, the difference set method will be employed.

## V. WHEEL PARTITION TECHNIQUE

In this section, we develop a novel technique, namely a wheel partition technique denoted by $WPT(v)$, to prove that cyclic triple factorization of order $v$ is $CTS(v,12)$. In addition, $WPT(v)$ will be utilized to formulate an algorithm for starter triples of $CTF(v)$.

The strategy of $WPT(v)$ for constructing $CTF(v)$ is divided into four steps as follows:

**Step 1.** Construct the starter of $CCF(4K_v,\delta)$.
**Step 2.** Generate wheel graphs by employing the cycles in Step 1
**Step 3.** Partition the wheel graphs in Step 2 into triples.
**Step 4.** Use the triples from Step 3 as a starter triples of $CTF(v)$ to enumerate all the triples by adding one modular $v$.

Fig. 3 shows the strategy of implementing the wheel partition technique on a set of cycles of $Z_9$ which satisfies a near-two-factor.

The wheel partition technique is exploited to demonstrate that the starter of $CTF(v)$ is a starter of a cyclic 12-fold triple system of order $v$ for $v = 12n+10$.

**Theorem 5.** For $v = 12n+10$, there exists a 12-fold cyclic triple factorization of order $v$.

**Proof.** We will prove this theorem by employing $WPT$ as follows:

**Step 1.** Construct the starter $\delta$ of $CCF(4K_{12n+10},\delta)$.

Consider $\delta = \{C_4^{3n+2}, C_{6n+5}^2\}$ is a starter set of $CCF(4K_{12n+10},\delta)$ as shown in Fig. 1 and 2. Then, the cycles of $\delta$ form a near-four-factor of $4K_{12n+10}$ with isolated zero integer. Moreover, the list of differences of $\delta$ covers each element in $Z_{6n+5}^*$ four times and the middle difference $6n+5$ occurs twice based on Theorem 3.

**Step 2.** Employ the cycles of $\delta = \{C_4^{3n+2}, C_{6n+5}^2\}$ in Step 1 to construct wheel graphs.

To do this, we will append the isolated vertex, zero integer, to each cycle in $\delta = \{C_4^{3n+2}, C_{6n+5}^2\}$ by connecting it to all vertices of each cycle in $\delta$. At this step, a set of wheels $\mathcal{W} = \{W_5^{3n+2}, W_{6n+6}^2\}$ will be represented as follows:

$$\begin{aligned} W_5^{3n+2} &= \{0 + C_{4_i}, \quad 1 \leq i \leq 3n+2\}, \\ W_{6n+6}^2 &= \{0 + C_{(6n+5)_i}, \quad 1 \leq i \leq 2\}. \end{aligned}$$

Furthermore, the list of differences from $\mathcal{W} =$

$\{W_5^{3n+2}, W_{6n+6}^2\}$ is calculated as

$$D(\mathcal{W}) = CD(W_i) \cup ID(W_i), \qquad W_i \in \mathcal{W}.$$

where

$CD(W_i) =$

$\left\{D\left(C_{4_i}\right) \cup D\left(C_{(5n+5)_j}\right), 1 \le i \le 3n+2,\ 1 \le j \le 2\right\}.$

$ID(W_i) =$

$\left\{D\left(K_{(1,4)_i}\right) \cup D\left(K_{(1,5n+5)_j}\right), 1 \le i \le 3n+2, 1 \le j \le 2\right\}.$

Since $\delta = \{C_4^{3n+2}, C_{6n+5}^2\}$ forms a near-four-factor, then the internal differences of $\mathcal{W}$, $ID(\mathcal{W})$, cover $Z_{6n+6}^*$ four times except the middle difference $6n+5$ occurs twice based on Lemma 2. Hence, from Step 1, it can be noticed that

$$CD(\mathcal{W}) = ID(\mathcal{W}).$$

Therefore, the list of differences of $\mathcal{W} = \{W_5^{3n+2}, W_{6n+6}^2\}$ covers $Z_{6n+6}^*$ eight times except the middle difference $6n+5$ occurs four times.

**Step 3.** Partition the wheel graphs of $\mathcal{W} = \{W_5^{3n+2}, W_{6n+6}^2\}$ in Step 2 into separated triangles (triples).

The generated triples from dividing of wheels graph will be formed by joining every two internal edges with an edge of the cycle that connected them.

According to the generated triangles at this phase, each internal edge of the wheels in $\mathcal{W} = \{W_5^{3n+2}, W_{6n+6}^2\}$ shall appear twice as the edges for generated triangles, whilst the edge of the associated cycles with the wheels in $\mathcal{W} = \{W_5^{3n+2}, W_{6n+6}^2\}$ will occur once.

Therefore, the differences list of generated triples possesses of the cycles differences $(CD)$ once whereas the internal differences $(ID)$ twice. From the Steps 1 and 2, the differences list of generated triples at this step covers every nonzero element in $Z_{6n+6}$ twelve times except the middle difference $6n+5$ appears six times. Thus, the set of generated triples is $(12n+10, 3, 12)$-difference family, then it is considered a starter triples of $CTS(12n+10, 12)$ based on Theorem 1.

**Step 4.** Generate all triples of $CTS(12n+10, 12)$.

To generate all the triples of $CTS(12n+10, 12)$, the starter triples will be placed in the first row and then repeated addition of 1 modular $12n+10$. □

As a consequence result from Theorem 5, we have the following corollary.

**Corollary 1.** For $v = 12n+10$. There exists a cyclic $(8K_v, \mathcal{W})$-design where $\mathcal{W}$ is a set of wheel graphs.

**Proof.** Let $\mathcal{W} = \{W_5^{3n+2}, W_{6n+6}^2\}$ be the set of wheel graphs of $8K_{12n+10}$ that constructed in Step 2 of Theorem 5, then $\mathcal{W}$ is a starter set of cyclic $(8K_{12n+10}, \mathcal{W})$-design based on the Theorem 3. □

The starter triples are the essential tool to construct the $CTF(v)$. Thus, the developing of the starter construction of $CTF(v)$ will be discussed in the next section.

## VI. ALGORITHM FOR STARTER TRIPLES OF $CTF(12n+10)$

In this section, the starter of $CTF(v)$ is formulated and developed by performing the wheel partition technique on the starter of $CCF(4K_{12n+10}, \delta)$.

Based on the starter construction of $CCF(4K_{12n+10}, \delta)$, we have two cases which depend on whether $n$ is odd or even. The process of generating the starter set of $CTF(12n+10)$ is demonstrated as follows:

**Case 1**. $n$ is odd.

Consider $\delta = \{C_4^{3n+2}, C_{6n+5}^2\}$ be the starter set of $CCF(4K_{12n+10}, \delta)$ which has been constructed in Fig. 1. Then, the wheel partition technique is implemented on $\delta = \{C_4^{3n+2}, C_{6n+5}^2\}$ to generate the starter triples of $CTF(12n+10)$ as shown in Fig. 4.

From Fig. 4, the generated triple from the wheels that associated with 4-cycles will be expressed as subsets below:

$S_1 = \left\{\left\{0, \frac{5n+3}{2}, \frac{19n+17}{2}\right\}, \left\{0, \frac{19n+17}{2}, \frac{7n+7}{2}\right\}, \left\{0, \frac{7n+7}{2}, \frac{17n+13}{2}\right\},\right.$
$\left.\left\{0, \frac{17n+13}{2}, \frac{5n+3}{2}\right\}\right\},$

$S_2 = \left\{\{0, i, 12n+10-i\}, \quad 1 \le i \le 3n+2, i \ne \frac{5n+3}{2}\right\},$

$S_3 =$
$\left\{\{0, 12n+10-i, 6n+5+i\}, 1 \le i \le 3n+2,\ i \ne \frac{5n+3}{2}\right\},$

$S_4 =$
$\left\{\{0, 6n+5-i, 6n+5+i\},\ 1 \le i \le 3n+2,\ i \ne \frac{5n+3}{2}\right\},$

$S_5 = \left\{\{0, 6n+5-i, i\}, 1 \le i \le 3n+2, i \ne \frac{5n+3}{2}\right\}.$

While, the produced triples from wheel associated with $C_{6n+5}^*$, will be expressed as subsets below

$S_6 = \left\{\{0, i+1, 6n+6-i\}, 1 \le i \le 2n+1\right\},$

$S_7 = \left\{\{0, i+2, 6n+6-i\}, 1 \le i \le 2n\right\},$

$S_8 = \left\{\{0, 9n+7+i, 9n+5-i\}, 1 \le i \le n-1\right\},$

$S_9 = \left\{\{0, 9n+8+i, 9n+5-i\}, 1 \le i \le n-1\right\},$

$S_{10} = \{\{0, 3n+3, 3n+5\}, \{0, 3n+4, 3n+5\}, \{0, 12n+9, 2\}, \{0, 4n+5, 3n+3\}, \{0, 3n+3, 9n+8\}, \{0, 10n+7, 12n+9\}\}.$

As shown above, the triples of $\{S_6 \cup S_7\}$ and $\{S_8 \cup S_9\}$ were generated by linking the edges of path $P_{4n+2}^*$ and path $P_{2n-1}^*$ of $C_{6n+5}^*$, respectively, with the isolated vertex $\{0\}$, while the set $S_{10}$ contained generated triples by linking the edges of $P_3^*$ with the isolated vertex $\{0\}$. Additionally, the triples are produced by joining the edges that connect of paths of $C_{6n+5}^*$ together with isolated $\{0\}$, along with linking the edges that connected the paths of $C_{6n+5}^*$ and the $e_0^* = 12n+9$ with isolated vertex $\{0\}$.

Similarly, it would be expressed of the generated triples from the wheel that associated with $C_{6n+5}^{**}$ as subsets below

$S_{11} = \left\{\{0, 12n+9-i, 6n+4+i\}, 1 \le i \le 2n+1\right\},$

$S_{12} = \left\{\{0, 12n+8-i, 6n+4+i\}, 1 \le i \le 2n\right\},$

$S_{13} = \left\{\{0, 3n+3-i, 3n+5+i\}, 1 \le i \le n-1\right\},$

$S_{14} = \left\{\{0, 3n+2-i, 3n+5+i\}, 1 \le i \le n-1\right\},$

$S_{15} = \{ \{0,1,12n+8\}, \{0,8n+5,9n+7\}, \{0,9n+6,3n+2\}, \{0,2n+3,1\}, \{0,9n+7,9n+5\}, \{0,9n+5,9n+6\}\}$

For the sake simplicity, the starter of cyclic triple factorization of order $12n+10$, can be represented as

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$$

such that

$$\mathcal{A}_1 = \begin{cases} \{0, i, 12n+10-i\}, 1 \le i \le 6n+4, \\ \{0, 12n+10-i, 6n+5+i\}, 1 \le i \le 3n+2, i \ne \frac{5n+3}{2}, \\ \{0, 6n+5-i, i\}, 1 \le i \le 3n+2, i \ne \frac{5n+3}{2}, \\ \{0, i+1, 6n+6-i\}, 1 \le i \le 2n+1, \\ \{0, 12n+9-i, 6n+4+i\}, 1 \le i \le 2n+1 \\ \{0, i+2, 6n+6-i\}, 1 \le i \le 2n, \\ \{0, 12n+8-i, 6n+4+i\}, 1 \le i \le 2n, \\ \{0, 9n+6+i, 9n+6-i\}, 1 \le i \le n, \\ \{0, 3n+4-i, 3n+4+i\}, \quad 1 \le i \le n, \\ \{0, 9n+8+i, 9n+5-i\}, 1 \le i \le n-1, \\ \{0, 3n+2-i, 3n+5+i\}, \quad 1 \le i \le n-1. \end{cases}$$

$\mathcal{A}_2 = \left\{\left\{0, \frac{19n+17}{2}, \frac{7n+7}{2}\right\}, \left\{0, \frac{7n+7}{2}, \frac{17n+13}{2}\right\}, \{0, 3n+4, 3n+5\}, \{0, 12n+9, 2\}, \{0, 4n+5, 3n+3\}, \{0, 3n+4, 9n+8\}, \{0, 10n+7, 12n+9\}, \{0, 1, 12n+8\}, \{0, 8n+5, 9n+7\}, \{0, 9n+6, 3n+2\}, \{0, 2n+3, 1\}, \{0, 9n+5, 9n+6\}\right\}$.

**Case 2.** $n$ is even.

Consider that $\delta = \{C_4^{3n+2}, C_{6n+5}^2\}$ is be the starter set of $CCF(4K_{12n+10}, \delta)$ which has been constructed in Fig. 2. Then, Fig. 5 shows the starter triples of $CTF(12n+10)$ by applying wheel partition technique on $\delta = \{C_4^{3n+2}, C_{6n+5}^2\}$.

From Fig. 5, all generated triples will be analyzed. We begin with produced triples from wheels that associated with 4-cycles as follows:

$S_1 = \left\{\left\{0, \frac{n}{2}, \frac{11n+10}{2}\right\}, \left\{0, \frac{11n+10}{2}, \frac{23n+20}{2}\right\}, \left\{0, \frac{23n+20}{2}, \frac{13n+10}{2}\right\}, \left\{0, \frac{13n+10}{2}, \frac{n}{2}\right\}\right\}$,

$S_2 = \left\{\{0, i, 12n+10-i\}, 1 \le i \le 3n+2, i \ne \frac{n}{2}\right\}$,

$S_3 = \left\{\{0, 12n+10-i, 6n+5+i\}, 1 \le i \le 3n+2, i \ne \frac{n}{2}\right\}$,

$S_4 = \left\{\{0, 6n+5-i, 6n+5+i\}, 1 \le i \le 3n+2, i \ne \frac{n}{2}\right\}$,

$S_5 = \left\{\{0, i, 6n+5-i\}, 1 \le i \le 3n+2, i \ne \frac{n}{2}\right\}$.

Note that the generated triples of $S_5$ and $S_3$ are

$S_3 = \{\{0, 12n+9, 6n+6\}, \{0, 12n+8, 6n+7\}, \ldots, \{0, 9n+8, 9n+7\}\} - \left\{0, \frac{23n+20}{2}, \frac{13n+10}{2}\right\}$.

$S_5 = \{\{0, 6n+4, 1\}, \{0, 6n+3, 2\}, \ldots, \{0, 3n+3, 3n+2\}\} - \left\{0, \frac{n}{2}, \frac{11n+10}{2}\right\}$.

Since $\left\{\left\{0, \frac{n}{2}, \frac{11n+10}{2}\right\}, \left\{0, \frac{23n+20}{2}, \frac{13n+10}{2}\right\}\right\} \in S_1$, then $S_3$ and $S_5$ could be represented as:

$S_3 = \left\{\{0, 12n+10-i, 6n+5+i\}, 1 \le i \le 3n+2\right\}$.

$S_5 = \left\{\{0, 6n+5-i, i\}, 1 \le i \le 3n+2\right\}$.

Clearly, it can be observed that the generated triples from the $(6n+5)$-cycles in Fig.5 are almost the same as generated triples in Fig. 4 with a slight difference. Fig. 6 shows the difference between the generated triples from $(6n+5)$-cycles in Fig. 4 and 5. Thus, we need to change some of triples as follows:

$$\{0, 3n+3, 4n+5\} \rightarrow \{0, 3n+5, 4n+5\}$$
$$\{0, 3n+4, 3n+5\} \rightarrow \{0, 3n+3, 3n+4\}$$
$$\{0, 9n+7, 8n+5\} \rightarrow \{0, 9n+5, 8n+5\}$$
$$\{0, 9n+6, 9n+5\} \rightarrow \{0, 9n+7, 9n+6\}$$

Therefore, the starter triples algorithm of the cyclic triple factorization of order $12n+10$ is formulated when $n$ is even as follows:

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$$

such that

$$\mathcal{A}_1 = \begin{cases} \{0, 12n+10-i, 6n+5+i\}, 1 \le i \le 3n+2, \\ \{0, 6n+5-i, i\}, 1 \le i \le 3n+2, \\ \{0, i, 12n+10-i\}, 1 \le i \le 3n+2, i \ne \frac{n}{2}, \\ \{0, 6n+5-i, 6n+5+i\}, 1 \le i \le 3n+2, i \ne \frac{n}{2}, \\ \{0, i+1, 6n+6-i\}, 1 \le i \le 2n+1, \\ \{0, 12n+9-i, 6n+4+i\}, \ 1 \le i \le 2n+1, \\ \{0, i+2, 6n+6-i\}, 1 \le i \le 2n, \\ \{0, 12n+8-i, 6n+4+i\}, 1 \le i \le 2n, \\ \{0, 9n+6+i, 9n+6-i\}, 1 \le i \le n, \\ \{0, 3n+4-i, 3n+4+i\}, 1 \le i \le n, \\ \{0, 9n+8+i, 9n+5-i\}, 1 \le i \le n-1, \\ \{0, 3n+2-i, 3n+5+i\}, 1 \le i \le n-1. \end{cases}$$

$\mathcal{A}_2 = \left\{\left\{0, \frac{11n+10}{2}, \frac{23n+20}{2}\right\}, \left\{0, \frac{13n+10}{2}, \frac{n}{2}\right\}, \{0, 3n+4, 3n+3\}, \{0, 12n+9, 2\}, \ \{0, 4n+5, 3n+5\}, \ \{0, 3n+4, 9n+8\}, \{0, 10n+7, 12n+9\}, \{0, 1, 12n+8\}, \{0, 8n+5, 9n+5\}, \{0, 9n+6, 3n+2\}, \{0, 2n+3, 1\}, \{0, 9n+6, 9n+7\}\right\}$.

**Example 2.** Based on the above algorithm of $CTF(12n+10)$ when $n$ is even, the starter triples $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ of $CTF(34)$ can be formed as follows:

$\mathcal{A}_1 = \{\{0,33,18\}, \{0,32,19\}, \{0,31,20\}, \{0,30,21\}, \{0,29,22\}, \{0,28,23\}, \{0,27,24\}, \{0,26,25\}\{0,16,1\}, \{0,15,2\}, \{0,14,3\}, \{0,13,4\}, \{0,12,5\}, \{0,11,6\}, \{0,10,7\}, \{0,9,8\}, \{0,2,32\}, \{0,3,31\}, \{0,4,30\}, \{0,5,29\}, \{0,6,28\}, \{0,7,27\}, \{0,8,26\}, \{0,15,19\}, \{0,14,20\}, \{0,13,21\}, \{0,12,22\}, \{0,11,23\}, \{0,10,24\}, \{0,9,25\}, \{0,2,17\}, \{0,3,16\}, \{0,4,15\}, \{0,5,14\}, \{0,6,13\}, \{0,32,17\}, \{0,31,18\}, \{0,30,19\}, \{0,29,20\}, \{0,28,21\}, \{0,3,17\}, \{0,4,16\}, \{0,5,15\}, \{0,6,14\}, \{0,31,17\}, \{0,30,18\}, \{0,29,19\}, \{0,28,20\}, \{0,25,23\}, \{0,26,22\}, \{0,9,11\}, \{0,8,12\}, \{0,27,22\}, \{0,7,12\}\}$.

$\mathcal{A}_2 = \{\{0,16,33\}, \{0,18,1\}, \{0,10,9\}, \{0,33,2\}, \{0,13,11\}, \{0,10,26\}, \{0,27,33\}, \{0,1,32\}, \{0,21,23\}, \{0,24,8\}, \{0,7,1\}, \{0,24,25\}\}$.

## VII. CONCLUSION

This article introduced a new type of cyclic triple system called cyclic triple factorization, $CTF(v)$, which satisfies some restrictions. Then, a new method, wheel partition technique, has been developed to prove that $CTF(v)$ represents a cyclic 12-fold triple system by exploiting cyclic $(C_4^{3n+2}, C_{6n+5}^2)$-cycle factorization of $4K_v$ when $v = 12n + 10$. Finally, the algorithms of the starter triples of $CTF(12n + 10)$ have been formulated. We expect the construction of $CTF(v)$ can be developed and extended for $v \equiv 2, 6 (mod\ 12)$.

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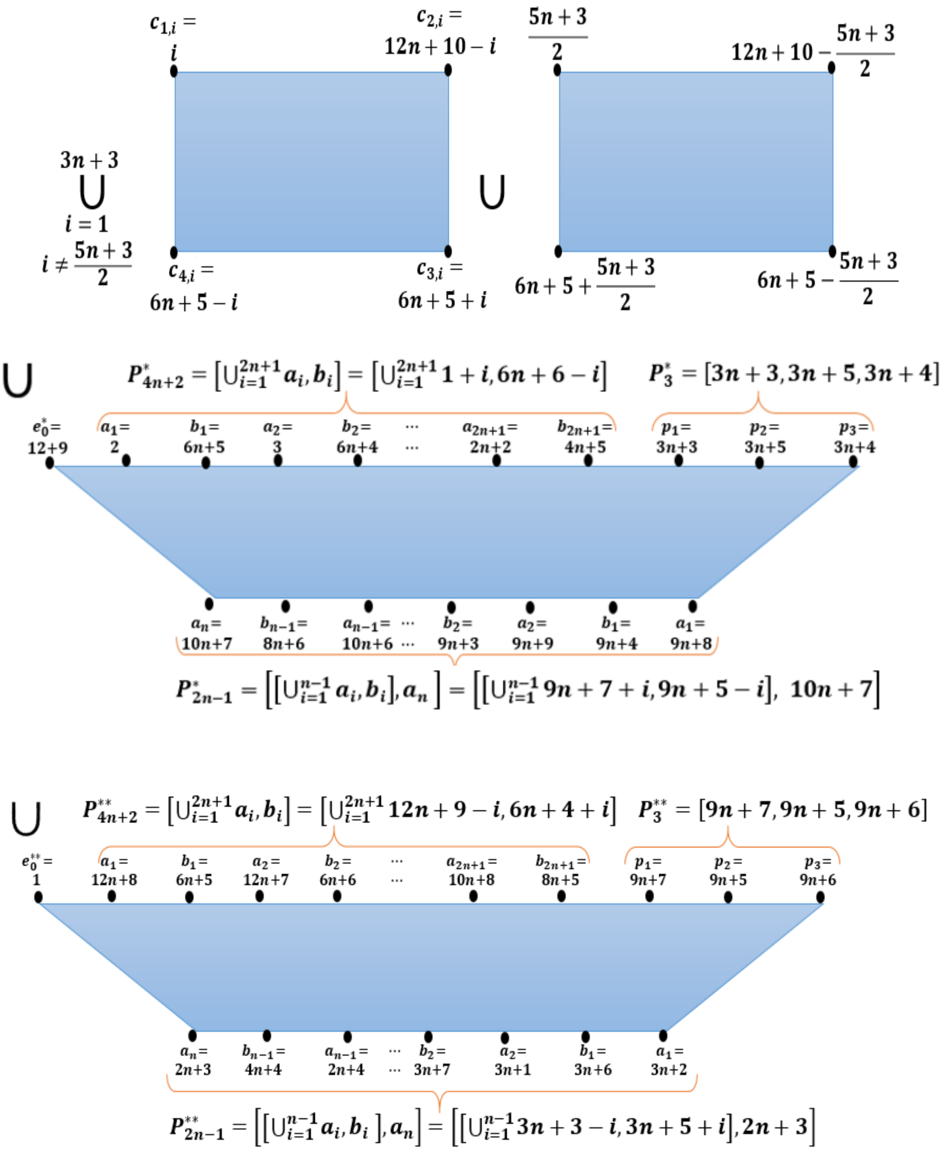

Fig. 1. Construction of cycles set $\delta = \{C_4^{3n+2}, C_{6n+5}^2\}$ of $4K_{12n+10}$, when $n$ is odd

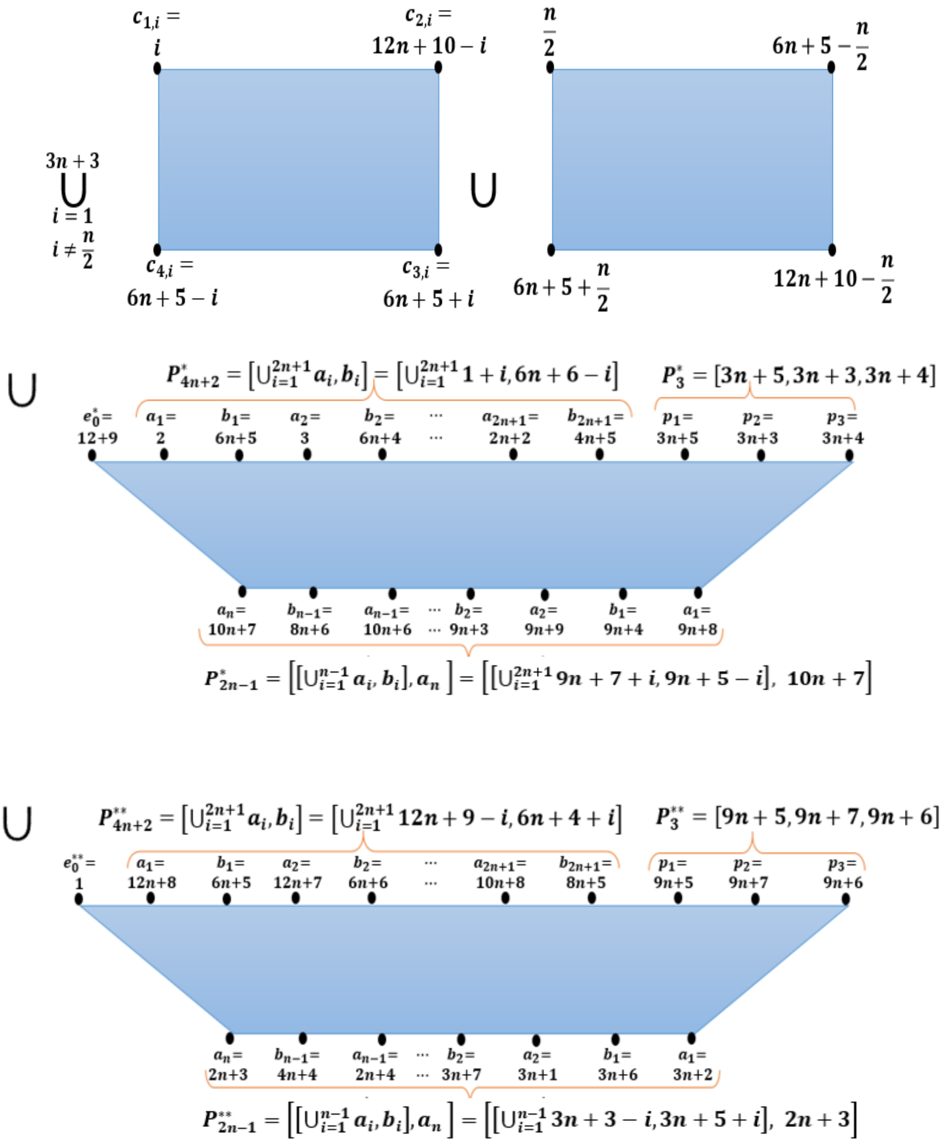

Fig 2. Construction of the cycles set $\delta = \{C_4^{3n+2}, C_{6n+5}^2\}$ of $4K_{12n+10}$, when $n$ is even

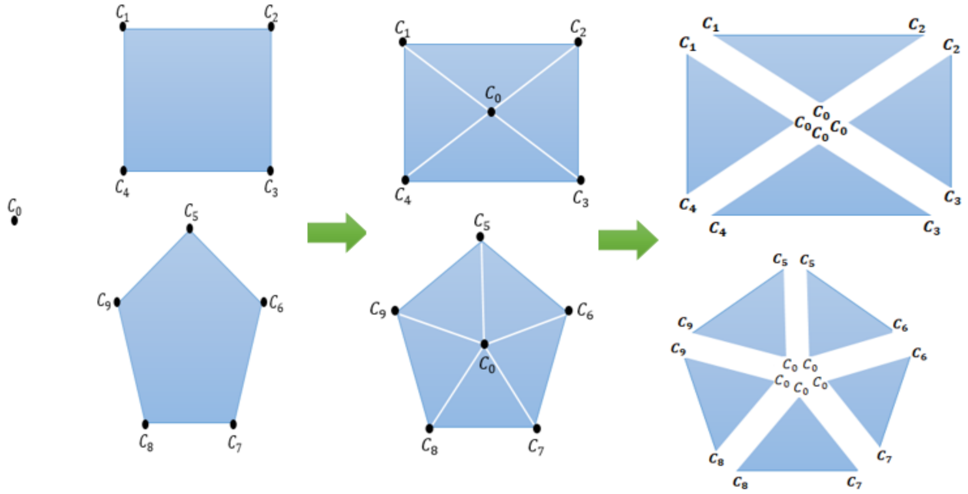

Fig. 3. Performing the wheel partition technique on a set of cycles.

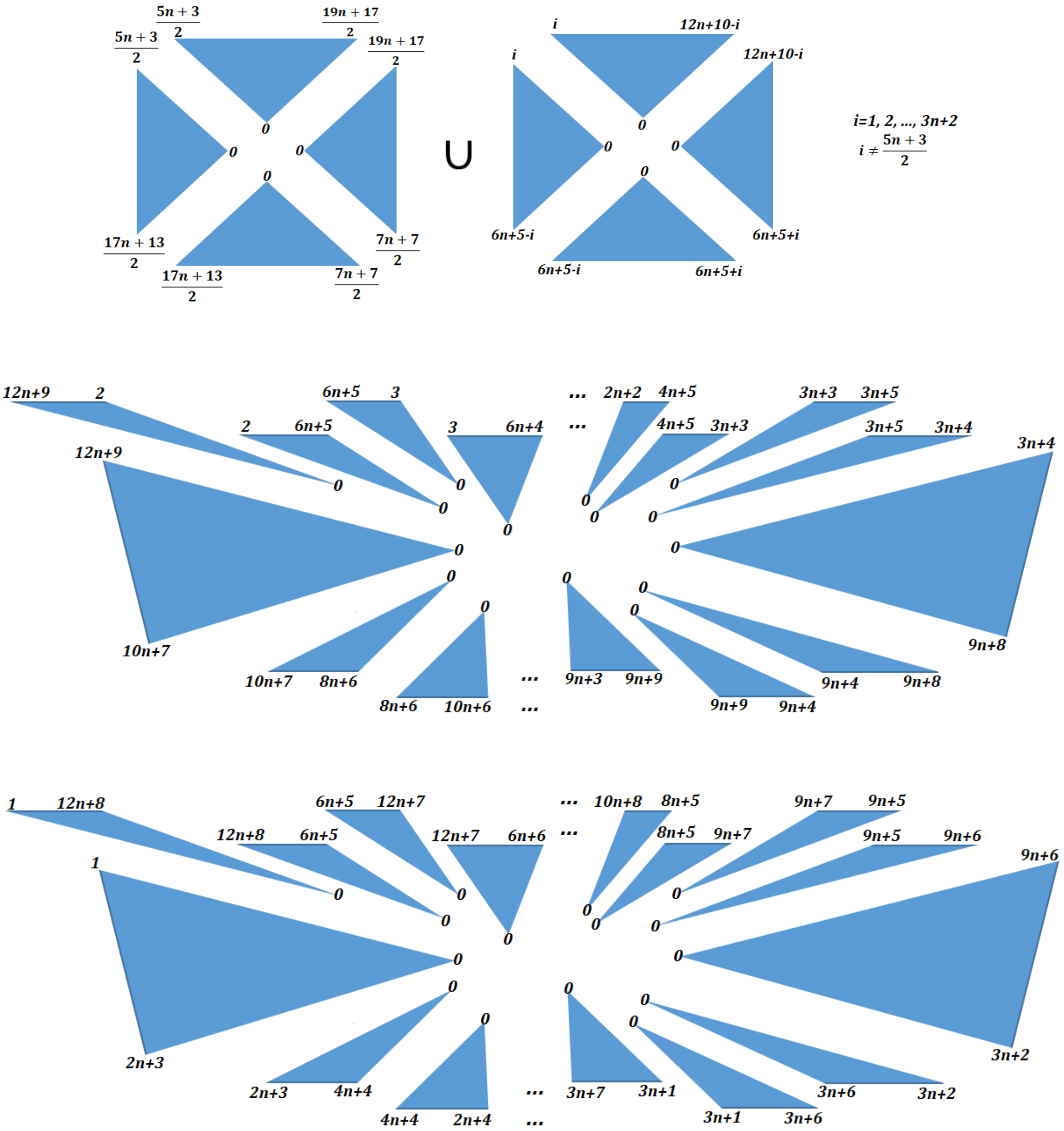

Fig. 4. Starter triples of $CTF(12n+10)$ when $n$ is odd

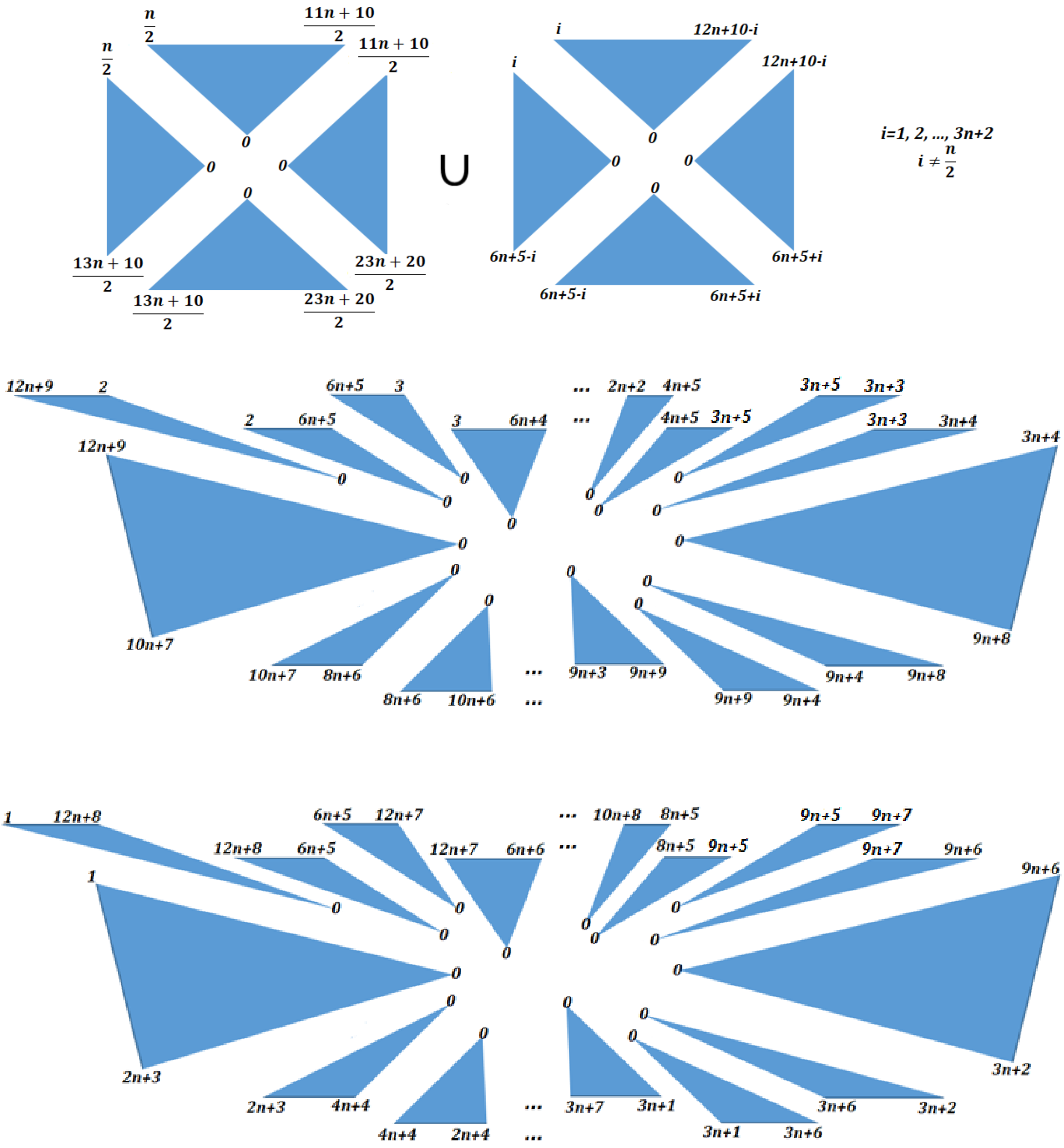

Fig. 5. Starter set of $CTF(12n+10)$ when $n$ is even

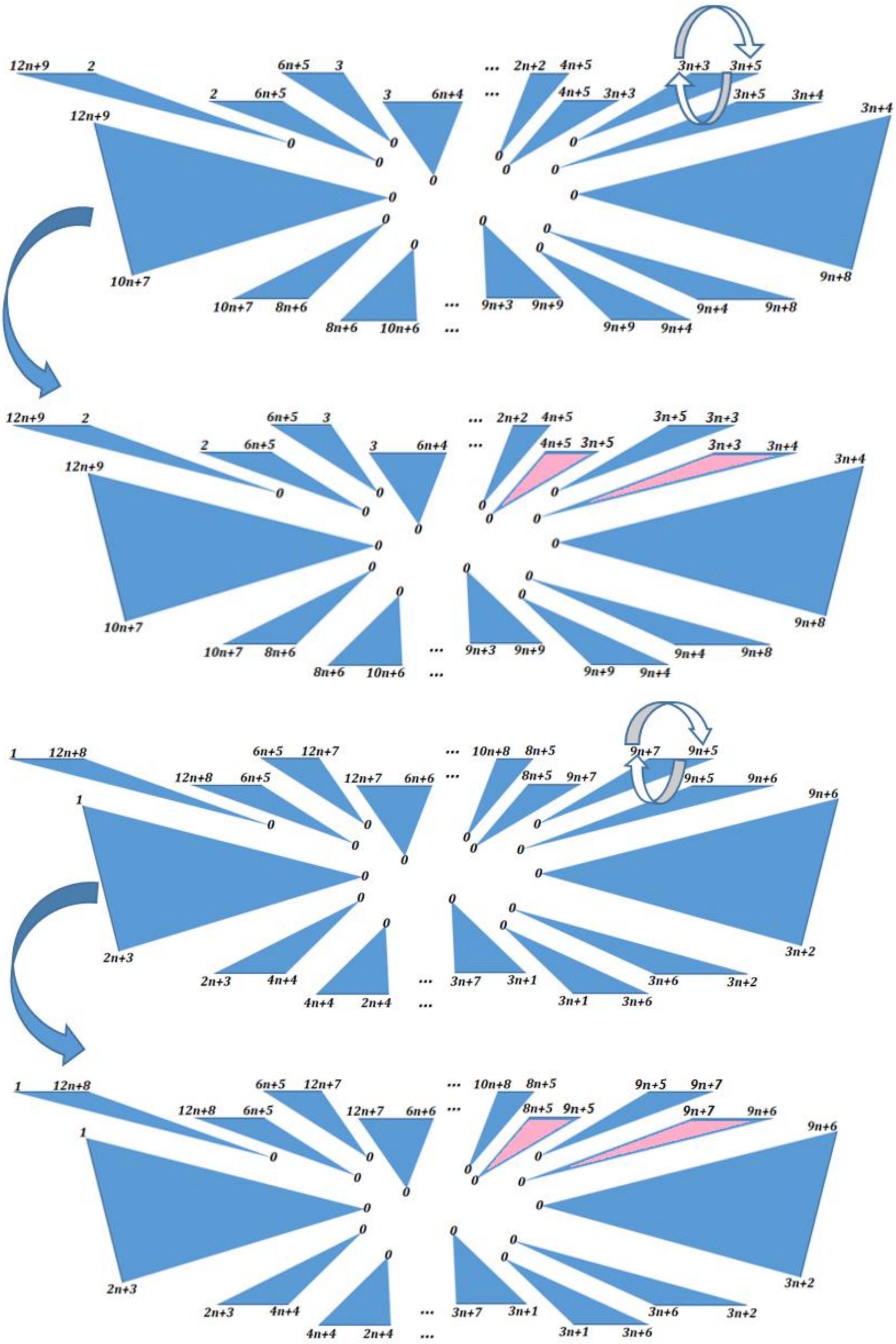

Fig. 6. The difference between the generated triples from the cycles of order $(6n+5)$ in Fig 4 and 5.