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Construction of Differential Operators

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Abstract

Due to its importance in generalizing and preserving subclasses of univalent functions, differential and integral operators have aroused a great interest in the theory of geometric functions. However, little attention has been paid to construct differential operators that preserve subclasses of univalent functions in comparison with integral operators who on many occasions constructed to do so. This paper discusses constructing differential operators by means of convolution, linear combination, and composition and raises some open questions in the direction of constructing differential operators that map analytic functions into univalent ones. Also, this paper raises questions regarding subclasses of univalent functions that preserve the univalent property under convolutions and linear combinations.

Keywords: *Complex function, analytic function, univalent functions, differential operators, Hadamard product(convolution), linear combinations, compositions, preservations.*

2020 Mathematical Subject Classification: 30C45.

1 Introduction

Throughout this paper, we are considering complex functions (i.e complex-valued function of complex variable) in the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and normalised by the conditions $f(0) = 0$ and $f'(0) = 1$. Even thought we may consider any type of complex function (e.g analytic, harmonic, meromorphic, etc.); we pick analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

to our consideration and refer to them by $\mathcal{A}$. Moreover, a subclass of $\mathcal{A}$ that has an additional property to be univalent is donated by $\mathcal{S}$.

There have been a significant number of differential operators produced. Many of them, without a doubt, are generic. However, just a few of them have been put together. In the same way, certain early differential operators and their power series extension for future construction are worth mentioning.

Ruscheweyh [15] defined the differential operator

$$R^{\alpha} : \mathcal{A} \to \mathcal{A} \tag{2}$$

where $\alpha \in \mathbb{N}_0 (\mathbb{N}_0 := \mathbb{N} \cup \{0\})$ and

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= z f'(z) \\ R^2 f(z) &= z f'(z) + \frac{1}{2} z^2 f''(z) \\ &\vdots \\ (\alpha + 1) R^{\alpha+1} f(z) &= \alpha R^{\alpha} f(z) + z (R^{\alpha} f(z))'. \end{aligned}$$

Now, if we write the analytic function f in terms of its series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

we get to

$$R^{\alpha} f(z) = \sum_{k=2}^{\infty} C(\alpha, k) a_k z^k, \tag{3}$$

where $C(\alpha, k) = \binom{k+\alpha-1}{\alpha}$.

Sălăgean [16] defined the following differential operator

$$S^n : \mathcal{A} \to \mathcal{A} \tag{4}$$

where $n \in \mathbb{N}_0$ and

$$S^0 f(z) = f(z)$$

$$S^1 f(z) = zf'(z)$$
$$\vdots$$
$$S^n f(z) = z(S^{n-1} f(z))'.$$

Writing the analytic function f in terms of its series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

yields to

$$S^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k. \tag{5}$$

Example 1.1. *Consider the function $\ell(z) = \frac{z}{1-z}$. Obviously, the function $\ell(z)$ is belong to $\mathcal{A}$ in $\mathbb{U}$. The Sălăgean's operator over $\ell(z)$ is obtained as follows:*

$$\begin{aligned} S^0 \ell(z) &= \ell(z) \\ S^1 \ell(z) &= z\ell'(z) = \frac{z}{(1-z)^2} \\ &\vdots \\ S^n \ell(z) &= \sum_{k=1}^{\infty} k^n z^k. \end{aligned}$$

By generalising Sălăgean operator, Al-Oboudi [6] introduced the following operator.

$$D_\lambda^n : \mathcal{A} \to \mathcal{A} \tag{6}$$

where $n \in \mathbb{N}_0, \lambda \geq 0$ and

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1 f(z) &= (1-\lambda) f(z) + zf'(z) = D_\lambda(z) = D_\lambda \\ &\vdots \\ D_\lambda^n f(z) &= D_\lambda(D^{n-1} f(z)). \end{aligned}$$

Writing the analytic function f in terms of its series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

yields to

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n a_k z^k. \tag{7}$$

Example 1.2. *Consider the function* $\ell(z) = \frac{z}{1-z} \in \mathcal{A}$ *in* $\mathbb{U}$. *Then*

$$\begin{aligned} D^0\ell(z) &= \ell(z) \\ D^1\ell(z) &= (1-\lambda)\ell(z) + z\ell'(z) \\ &\vdots \\ D_\lambda^n\ell(z) &= \sum_{k=1}^{\infty} [1+\lambda(k-1)]^n z^k. \end{aligned}$$

There have been a slew of other differential operators developed and generalised, to name a few [1-5] and [7].

The following three sections elaborate three different ways to construct differential operators. Explicitly, these ways are convolution, linear combination, and composition. Later in Section 5, we discuss the preservation of univalent condition under convolution, linear combination, and composition. Section 6 discusses conditions that preserve univalence property of certain function. Finally, Section 7 concludes the paper.

2 Convolutions of Differential Operators

Convolution can be used to express many operators. This is evident from the definition of convolution, which allows for the splitting of coefficients. Ruscheweyh's operator, on the other hand, has a different interpretation.

$$R^\alpha f(z) = f(z) * \frac{z}{(1-z)^{\alpha+1}}, \qquad \alpha > -1,$$

which implies that

$$R^n f(z) = \frac{z\left(z^{n-1}f(z)\right)^{(n)}}{n!} \qquad n = 1, 2, 3\ldots.$$

Example 2.1. *Consider the function* $\ell(z) = \frac{z}{1-z} \in \mathcal{A}$ *in* $\mathbb{U}$. *Then*

$$\begin{aligned} R^0\ell(z) &= \ell(z) \\ R^1\ell(z) &= z\ell'(z) \\ &\vdots \\ R^\alpha\ell(z) &= \sum_{k=1}^{\infty} C(\alpha,k)z^k. \end{aligned}$$

Notice that, expressing the operator D_λ^n defined in (7) in terms of convolution can be made by

$$D_\lambda^n f(z) = \underbrace{\varphi(z) * \cdots * \varphi(z)}_{n-times} * f(z)$$

where

$$\varphi(z) = \frac{z}{1-z} + \frac{\lambda z}{(1-z)^2} - \frac{\lambda z}{1-z}.$$

Darus and Al-Shaqsi proposed the differential operator in [9] as follows.

$$R_{\alpha,\lambda}^n : \mathcal{A} \to \mathcal{A}$$

where $\lambda \geq 0$, and n, $\alpha \in \mathbb{N}_0$. and

$$\begin{aligned}
R_{\alpha,\lambda}^0 &= f(z) \\
R_{\alpha,\lambda}^1 &= zf'(z) + \lambda z^2 f''(z) = R^* \\
&\vdots \\
R_{\alpha,\lambda}^n f(z) &= R^* \left(R_{\alpha,\lambda}^{n-1} f(z) \right).
\end{aligned}$$

Writing the analytic function f in terms of its series expansion $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, yields to

$$R_{\alpha,\lambda}^n f(z) = z + \sum_{k=2}^{\infty} [1 + \beta(k-1)]^n C(\alpha, k)\ a_k z^k. \tag{8}$$

In terms of convolution, $R_{\alpha,\lambda}^n$ may be rewritten as

$$R_{\alpha,\lambda}^n = \underbrace{\varphi(z) * \cdots * \varphi(z)}_{n-times} * \frac{z}{(1-z)^{\alpha+1}} * f(z).$$

Remark 2.1. *The following table provides the power series and convolution forms for each of R^n, S^n, D_λ^n and $R_{\alpha,\lambda}^n$.*

Yet in this paper, there are no two existence differential operator have been convoluted. In [12] Lupas considered the differential operator SR_α^n which is the convolution of S^n and R^α. More precisely,

$$\begin{aligned}
SR_\alpha^n f(z) &= S^n f(z) * R^\alpha f(z) \\
&= \left(z + \sum_{k=2}^{\infty} k^n a_k z^k \right) * \left(z + \sum_{k=2}^{\infty} C(\alpha, k)\ a_k z^k \right) \\
&= z + \sum_{k=2}^{\infty} k^n C(\alpha, k)\ a_k^2 z^k.
\end{aligned}$$

Power Series Form	Convolution Form
$R^n f(z) = z + \sum_{k=2}^{\infty} C(\alpha, k) a_k z^k$	$R^n f(z) = \frac{z}{(1-z)^{\alpha+1}} * f(z)$
$S^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k$	$S^n f(z) = \underbrace{\kappa(z) * \cdots * \kappa(z)}_{n-times} * f(z)$
$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} \left[1 + \lambda(k-1)\right]^n a_k z^k$	$D_\lambda^n f(z) = \underbrace{\varphi(z) * \cdots * \varphi(z)}_{n-times} * f(z)$
$R_{\alpha,\lambda}^n f(z) = z + \sum_{k=2}^{\infty} [1 + \beta(k-1)]^n C(\alpha, k)\ a_k z^k$	$R_{\alpha,\lambda}^n = \underbrace{\varphi(z) * \cdots * \varphi(z)}_{n-times} * \frac{z}{(1-z)^{\alpha+1}} * f(z)$

Table 1: Differential Operators.

Andrei took into account the differential operator $DR_{\alpha,\lambda}^n$, which is the convolution of D_λ^n and R^α, in [8]. Specifically,

$$\begin{aligned} DR_{\alpha,\lambda}^n f(z) &= D_\lambda^n f(z) * R^\alpha f(z) \\ &= \left(z + \sum_{k=2}^{\infty} \left[1 + \lambda(k-1)\right]^n a_k z^k\right) * \left(z + \sum_{k=2}^{\infty} C(\alpha, k)\ a_k z^k\right) \\ &= z + \sum_{k=2}^{\infty} \left[1 + \lambda(k-1)\right]^n C(\alpha, k)\ a_k^2 z^k. \end{aligned}$$

The differential operators $R_{\alpha,\lambda}^n$ and $DR_{\alpha,\lambda}^n$ appear to be very similar at first glance, but they are actually considerably distinct in terms of their construction and even their coefficients. That is, in general, $R_{\alpha,\lambda}^n$ and $DR_{\alpha,\lambda}^n$ are not equal. The following example will demonstrate the distinction.

Example 2.2. *The Koebe function $\kappa(z) = \frac{z}{(1-z)^2}$ under the convoluted differential operator $DR_{\alpha,\lambda}^n$ takes the following manner:*

$$\begin{aligned} DR_{\alpha,\lambda}^n \kappa(z) &= D_\lambda^n \kappa(z) * R^\alpha \kappa(z) \\ &= \left(z + \sum_{k=2}^{\infty} \left[1 + \lambda(k-1)\right]^n k^n z^k\right) * \left(z + \sum_{k=2}^{\infty} C(\alpha, k)\ k^n z^k\right) \\ &= z + \sum_{k=2}^{\infty} \left[1 + \lambda(k-1)\right]^n C(\alpha, k)\ k^{2n} z^k. \end{aligned}$$

Under the differential operator $R_{\alpha,\lambda}^n$, on the other hand, $\kappa(z)$ behaves dif-

ferently as follows.

$$\begin{aligned}
R^0_{\alpha,\lambda}\kappa(z) &= \kappa(z)\\
R^1_{\alpha,\lambda}\kappa(z) &= z\kappa'(z) + \lambda z^2\kappa''(z) = R^*\\
&\vdots\\
R^n_{\alpha,\lambda}\kappa(z) &= R^*\left(R^{n-1}_{\alpha,\lambda}\kappa(z)\right)\\
&= z + \sum_{k=2}^{\infty}\left[1+\lambda(k-1)\right]^n C(\alpha,k)\ z^k.
\end{aligned}$$

In some cases, however, the differential operators $R^n_{\alpha,\lambda}$ and $DR^n_{\alpha,\lambda}$ can be mapped onto the same function. Consider $\ell(z) = \frac{z}{1-z}$. It is clear that

$$R^n_{\alpha,\lambda}\ell(z) = DR^n_{\alpha,\lambda}\ell(z) = z + \sum_{k=2}^{\infty}\left[1+\lambda(k-1)\right]^n C(\alpha,k)\ z^k.$$

The differential operators $R^n_{\alpha,\lambda}$ and D^n_λ were used to generate another convoluted differential operator in [4]. Both of them have a convoluted form as follows.

$$\begin{aligned}
\tilde{D}^n_{\alpha,\lambda}f(z) &= D^n_\lambda f(z) * R^n_{\alpha,\lambda}f(z)\\
&= \left(z + \sum_{k=2}^{\infty}\left[1+\lambda(k-1)\right]^n a_k z^k\right) * \left(z + \sum_{k=2}^{\infty}\left[1+\lambda(k-1)\right]^n C(\alpha,k)\ a_k z^k\right)\\
&= z + \sum_{k=2}^{\infty}\left[1+\lambda(k-1)\right]^{2n} C(\alpha,k)\ a_k^2 z^k.
\end{aligned}$$

Example 2.3. *Again, we consider the extremal functions $\kappa(z)$ and $\ell(z)$ of univalent and convex classes respectively.*

$$\begin{aligned}
\tilde{D}^n_{\alpha,\lambda}\kappa(z) &= D^n_\lambda \kappa(z) * R^n_{\alpha,\lambda}\kappa(z)\\
&= \left(z + \sum_{k=2}^{\infty}\left[1+\lambda(k-1)\right]^n k^n z^k\right) * \left(z + \sum_{k=2}^{\infty}\left[1+\lambda(k-1)\right]^n C(\alpha,k)\ k^n z^k\right)\\
&= z + \sum_{k=2}^{\infty}\left[1+\lambda(k-1)\right]^{2n} C(\alpha,k)\ k^{2n} z^k.
\end{aligned}$$

and

$$
\begin{aligned}
\tilde{D}^n_{\alpha,\lambda}\ell(z) &= D^n_\lambda \kappa(z) * R^n_{\alpha,\lambda}\ell(z) \\
&= \left(z + \sum_{k=2}^{\infty} [1+\lambda(k-1)]^n z^k \right) * \left(z + \sum_{k=2}^{\infty} [1+\lambda(k-1)]^n C(\alpha,k)\ z^k \right) \\
&= z + \sum_{k=2}^{\infty} [1+\lambda(k-1)]^{2n} C(\alpha,k)\ z^k.
\end{aligned}
$$

The following proposition provides the $(n+1)$-derivative.

Proposition 2.1. *For* $n, \alpha \in \mathbb{N}$ *and* $\lambda \geq 0$ *we have*

$$
\tilde{D}^{n+1}_{\alpha,\lambda} f(z) = -2\lambda \tilde{D}^n_{\alpha,\lambda} f(z) - 2(\lambda+1) z \left(\tilde{D}^n_{\alpha,\lambda} f(z)\right)' + z^2 \left(\tilde{D}^n_{\alpha,\lambda} f(z)\right)''
$$

and

$$
z\left(\tilde{D}^n_{\alpha,\lambda} f(z)\right)' = (\alpha+1)\tilde{D}^n_{\alpha+1,\lambda} f(z) - \alpha \tilde{D}^n_{\alpha,\lambda} f(z).
$$

Proof. We have

$$
\begin{aligned}
\tilde{D}^{n+1}_{\alpha,\lambda} f(z) &= z + \sum_{k=2}^{\infty} [1+\lambda(k-1)]^{2(n+1)} C(\alpha,k) k^{2n} z^k \\
&= z + \sum_{k=2}^{\infty} [1+\lambda(k-1)]^2 [1+\lambda(k-1)]^{2n} C(\alpha,k) a_k^2 z^k \\
&= z + \sum_{k=2}^{\infty} \left[k^2 - 2k[\lambda+1] - 2\lambda\right] [1+\lambda(k-1)]^{2n} C(\alpha,k)\ a_k^2 z^k \\
&= -2\lambda \tilde{D}^n_{\alpha,\lambda} f(z) - 2(\lambda+1) z \left(\tilde{D}^n_{\alpha,\lambda} f(z)\right)' + z^2 \left(\tilde{D}^n_{\alpha,\lambda} f(z)\right)''
\end{aligned}
$$

and

$$
\begin{aligned}
&(\alpha+1)\tilde{D}^{\alpha+1}_{\alpha,\lambda}f(z)-\alpha\tilde{D}^{n}_{\alpha,\lambda}f(z)\\
&=(\alpha+1)z+(\alpha+1)\sum_{k=2}^{\infty}[1+\lambda(k-1)]^{n}\,C(\alpha+1,k)a_k^2z^k\\
&-\alpha z-\alpha\sum_{k=2}^{\infty}[1+\lambda(k-1)]^{n}\,C(\alpha,k)a_k^2z^k\\
&=z+(\alpha+1)\sum_{k=2}^{\infty}[1+\lambda(k-1)]^{n}\,\frac{\alpha+k}{\alpha+1}C(\alpha,k)a_k^2z^k\\
&-\alpha\sum_{k=2}^{\infty}[1+\lambda(k-1)]^{n}\,C(\alpha,k)a_k^2z^k\\
&=z+\sum_{k=2}^{\infty}[1+\lambda(k-1)]^{n}\,C(\alpha,k)ka_k^2z^k\\
&=z\left(\tilde{D}^{n}_{\alpha,\lambda}f(z)\right)'
\end{aligned}
$$

□

We turn now to the second way, in this paper, of constructing differential operators.

3 Linear Combination of Differential Operators

Recently, many differential operators constructed by means of *linear combination.* We begin by mentioning the following constructed differential operators by Lupas [13]:

$$SR^n_\lambda f(z)=(1-\gamma)R^\alpha f(z)+\gamma S^n f(z),\quad z\in\mathbb{U}. \tag{9}$$

The differential operator SR^n_λ has been studied intensively by Lupas. Also Lupas [14], introduced another differential operator by means of linear combination

$$RD^n_\lambda f(z)=(1-\gamma)R^\alpha f(z)+\gamma D^n_\lambda f(z),\quad z\in\mathbb{U} \tag{10}$$

and studied several properties of functions involving it.

In [3], a differential operator was considered by means of linear combination of both $R^n_{\alpha,\lambda}$ and D^n_λ as follows:

$$D^n_{\lambda,\alpha,\gamma}f(z)=(1-\gamma)R^n_\alpha f(z)+\gamma D^n_\lambda f(z),\quad z\in\mathbb{U}. \tag{11}$$

Writing the analytic function f in terms of its series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

yields to

$$D^n_{\lambda,\alpha,\gamma} f(z) = z + \sum_{k=2}^{\infty} \Psi(n,k,\lambda,\gamma) a_k z^k, \tag{12}$$

where

$$\Psi(n,k,\lambda,\gamma) = [1 + \lambda(k-1)]^n \left[\gamma + (1-\gamma)\, C(\alpha,k)\right]. \tag{13}$$

The following proposition provides the $(n+1)$-derivative.

Proposition 3.1. *For* $n, \alpha \in \mathbb{N}$ *and* $\lambda, \gamma \geq 0$ *we have*

$$D^{n+1}_{\lambda,\alpha,\gamma} f(z) = D^n_{\lambda,\alpha,\gamma} - \lambda D^n_{\lambda,\alpha,\gamma} + \lambda z \left(D^n_{\lambda,\alpha,\gamma} f(z)\right)'$$

Proof. We have

$$\begin{aligned} D^{n+1}_{\lambda,\alpha,\gamma} f(z) &= z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^{n+1} \left[\gamma + (1-\gamma)\, C(\alpha,k)\right] a_k z^k \\ &= z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)][1 + \lambda(k-1)]^n \left[\gamma + (1-\gamma)\, C(\alpha,k)\right] a_k z^k \\ &= D^n_{\lambda,\alpha,\gamma} - \lambda D^n_{\lambda,\alpha,\gamma} + \lambda z \left(D^n_{\lambda,\alpha,\gamma} f(z)\right)'. \end{aligned}$$

□

We turn now to the third way, in this paper, of constructing differential operators.

4 Composition of Differential Operators

There are no much, as far as we know, of constructed differential operators by means of composition. Let us consider again the R^α and S^n. Now by take the composition of both of them we obtain

$$\begin{aligned} SR^n_\alpha f(z) &= S^n f(z) \circ R^\alpha f(z) \\ &= \left(z + \sum_{k=2}^{\infty} k^n a_k z^k\right) \circ \left(z + \sum_{k=2}^{\infty} C(\alpha,k)\ a_k z^k\right) \\ &= z + \sum_{k=2}^{\infty} C(\alpha,k)\ a_k z^k + \sum_{k=2}^{\infty} k^n\ a_k \left(z + \sum_{k=2}^{\infty} C(\alpha,k)\ a_k z^k\right)^k. \end{aligned}$$

This kind of operator might help to take a first step to investigate the subordination of differential operators rather of complex functions. The subordination of f and g is in fact defined by means of composing f with a Schwarz function w, that is,

$$g(z) = (f \circ w)(z) = f(w(z)), \qquad |z| < 1,$$

where w is an analytic function with $w(0) = 0$ and $|w(z)| \leq |z|$. This shows the crucial role of defining the composition operation between two differential operators; in order to study the subordination relations between differential operators.

It is worth mentioning that, composing differential operator with integral operator was explored in [17]. However, this section is regarding composing two differential operators.

5 Preservation Properties

In this section we are taking an account to study preservation. It is known that two univalent functions are not necessarily univalent under convolution nor under linear combination. But it is preserved under composition. Let us consider some example. If we consider Koebe function twice, then the convolution of both of them is

$$\begin{aligned}\kappa(z) * \kappa(z) &= \left(z + \sum_{k=2}^{\infty} kz^k\right) * \left(z + \sum_{k=2}^{\infty} kz^k\right)\\ &= z + \sum_{k=2}^{\infty} k^2 z^k.\end{aligned}$$

However any function in $\mathcal{S}$ has to have coefficients bound less than or equal k for all k (the proof of this result is in [10]). Thus, it is obvious that $\kappa(z) * \kappa(z)$ is not univalent since it has a coefficient k^2 which is greater than k for all k. Also, Examples 2.2 and 2.3 show dramatically how convolution does not preserve the univalent property when the case taken under operator.

Linear combination also does not preserve univalent property. For example, the functions

$$\ell(z) = \frac{z}{1-z} \qquad \text{and} \qquad h(z) = \frac{z}{1+iz}$$

are both univalent in $\mathbb{U}$; however, their linear combination is not. For example, making the linear combination of ℓ and h with $\gamma = 1/2$ yields to

$$\begin{aligned}g(z) &= (\gamma)\,\frac{z}{1-z} + (1-\gamma)\,\frac{z}{1+iz}\\ &= \left(\frac{1}{2}\right)\frac{z}{1-z} + \left(1 - \frac{1}{2}\right)\frac{z}{1+iz}.\end{aligned}$$

However, g' will vanish at $z = (1+i)/2$. Therefore, g is not univalent in $\mathbb{U}$. If we consider $D^n_{\lambda,\alpha,\gamma}\ell(z)$ and $D^n_{\lambda,\alpha,\gamma}h(z)$ for $\gamma = 1/2$ then the linear combination is

$$D^n_{\lambda,\alpha,\gamma}g(z) = \left(z + \sum_{k=2}^{\infty}\Psi(n,k,\lambda,\gamma)a_k z^k + z + \sum_{k=2}^{\infty}(-1)i^k\Psi(n,k,\lambda,\gamma)a_k z^k\right)$$

which is obviously not univalent in $\mathbb{U}$.

Composition *does* preserve univalent property for two univalent functions. Moreover, and from other point of view, it is worth to mention that the idea of composing a subclass of starlike functions, as a subclass of the class of univalent functions, to be starlike, was investigated in [11].

6 Results and Discussions

This section discusses conditions that preserve univalent property of the following complex function: $f(z) = z + a_2z^2$. Firstly, we discuss the univalence of $f(z) = z + a_2z^2$ in $\mathbb{U}$ and then we discuss the univalence of $S^n f(z)$.

Let $z_1, z_2 \in \mathbb{U}$ and suppose $f(z_1) = f(z_2)$. This means that

$$\begin{aligned} f(z_1) &= f(z_2) \\ \Rightarrow z_1 + a_2z_1^2 &= z_2 + a_2z_2^2 \\ \Rightarrow z_1 - z_2 + a_2(z_1^2 - z_2^2) &= 0 \\ \Rightarrow (z_1 - z_2)(1 + a_2(z_1 + z_2)) &= 0 \end{aligned}$$

that is, $f(z)$ is univalent if $|a_2| \leq \frac{1}{2}$.

Now, for $f(z) = z + a_2z^2$ we find $S^n f(z)$, that is Sălăgean differential operator over $f(z)$, as follows

$$\begin{aligned} S^0 f(z) &= f(z) \\ &= z + a_2z^2 \\ S^1 f(z) &= zf'(z) \\ &= z(z + a_2z^2)' \\ &= z + 2a_2z^2 \\ S^2 f(z) &= z(z + 2a_2z^2)' \\ &= z(1 + 4a_2z)' \\ &= z + 4a_2z^2 \\ &\vdots \end{aligned}$$

$$S^n f(z) = z + (2)^n a_2 z^2.$$

Following the same steps to cover *when* $f(z)$ is univalent, we find that $S^n f(z)$ is univalent if and only if $|a_2| \leq \frac{1}{(2)^{n+1}}$.

7 Open questions

The construction of differential operators leads to think about what kind of differential operator can be constructed to map analytic functions to univalent functions. In other words, it is worth to construct a differential operators, say D^n, such that

$$D^n : \mathcal{A} \to \mathcal{S}.$$

This, for instant, will help to proof the isomorphism, monomorphism, and homeomorphisms over the space of complex valued function. Sure, such constructed differential operator provides other technique for other branches of mathematics and even for other sciences dealing with complex valued functions.

On the other hand, what are the subclasses of $\mathcal{S}$ that their convolution is in $\mathcal{S}$? More precisely, we hope to find a such class $\mathcal{S}_1$ of $\mathcal{S}$ such that

$$f * g \in \mathcal{S}$$

where f and g are belong to $\mathcal{S}_1$.

Similarly, we hope to find a subclass, say $\mathcal{S}_2$, of $\mathcal{S}$ such that

$$\gamma f + (1-\gamma) g \in \mathcal{S}$$

where $f, g \in \mathcal{S}$.

8 Conclusions

Following the flow of creating differential operators, this paper has provided and emphasized three ways of constructing differential operators and some open questions in the direction of constructing differential operators that map analytic functions into univalent ones. Also, this paper has raised questions regarding subclasses of univalent functions that preserve univalence condition under convolutions, and linear combinations. In addition, several examples have been given. As well as, providing the univalence condition of $f(z) = z + a_2 z^2$ mapped by Sălăgean differential operator, $S^n f(z)$.

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