

See discussions, stats, and author profiles for this publication at: <https://www.researchgate.net/publication/324558875>

Purity Temperature Dependent for Coupled Harmonic Oscillators

Article in *Journal of Statistical Mechanics Theory and Experiment* · April 2018

DOI: 10.1088/1742-5468/aad19b

CITATIONS

0

READS

74

4 authors:



Merdaci Abdeldjalil
King Faisal University

24 PUBLICATIONS 63 CITATIONS

[SEE PROFILE](#)



Ahmed Jellal
Chouaib Doukkali University

191 PUBLICATIONS 867 CITATIONS

[SEE PROFILE](#)



Ayman Al Sawalha
Jerash University

16 PUBLICATIONS 163 CITATIONS

[SEE PROFILE](#)



Abdelhadi Bahaoui
Université Chouaib Doukkali

11 PUBLICATIONS 58 CITATIONS

[SEE PROFILE](#)

Some of the authors of this publication are also working on these related projects:



Quantum Hall Effect [View project](#)



Graphene Superlattice [View project](#)

Purity Temperature Dependent for Coupled Harmonic Oscillators

Abdeldjalil Merdaci^{*a}, Ahmed Jellal^{†b,c}, Ayman Al Sawalha^d and Abdelhadi Bahaoui^b

^a*Physics Department, College of Science, King Faisal University,
PO Box 380, Alahsa 31982, Saudi Arabia*

^b*Saudi Center for Theoretical Physics, Dhahran, Saudi Arabia*

^c*Theoretical Physics Group, Faculty of Sciences, Chouaib Doukkali University,
PO Box 20, 24000 El Jadida, Morocco*

^d*Physics Department, Faculty of Science, Jerash University, Jerash, Jordan*

Abstract

We consider the thermal aspect of a system composed of two coupled harmonic oscillators and study the corresponding purity. We initially consider a situation where the system is brought to a canonical thermal equilibrium with a heat-bath at temperature T . We adopt the path integral approach and introduce the evolution operator to calculate the density matrix and subsequently the reduced matrix density. It is used to explicitly determine the purity in terms of different physical quantities and therefore study some limiting cases related to temperature as well as other parameters. Different numerical results are reported and discussed in terms of the involved parameters of our system.

PACS numbers: 03.65.Ud, 03.65.-w, 03.67.-a

Keywords: Two coupled harmonic oscillator, path integral, density matrix, purity.

*amerdaci@kfu.edu.sa

†a.jellal@ucd.ac.ma

1 Introduction

For a given many body system, it is not easy to measure the quantum mechanical correlations because of their infinitesimal sizes. In some cases for instance like superconductivity, there exists strong correlations and the pure states persist even in the large number of the many body state. In such states the classical thermodynamic quantities have been theoretically defined and identified with measured variables but the quantum mechanical correlations are lost. One way to recover what were lost is to introduce the so-called entanglement as a mathematical tool to describe the correlated systems. Entanglement is one of the most remarkable features of quantum mechanics that does not have any classical counterpart. It is a notion which has been initially introduced and coined by Schrödinger [1] when quantum mechanics was still in its early stage of development. Its status has evolved throughout the decades and has been subjected to significant changes.

Traditionally, entanglement has been related to the most quantum mechanical exotic concepts such as Schrödinger cat [1], Einstein-Podolsky-Rosen paradox [2] and violation of Bell's inequalities [3]. Despite its conventional significance, entanglement has gained, in the last decades, a renewed interest mainly because of the development of the quantum information science [4]. It has been revealed that it lies at the heart of various communication and computational tasks that cannot be implemented classically. It is believed that entanglement is the main ingredient of the quantum speed-up in quantum computation [4]. Moreover, several quantum protocols such as teleportation, quantum dense coding, and so on [5–11] are exclusively realized with the help of entangled states.

We propose a new approach to explicitly determine the purity function for a given system for the whole energy spectrum rather than the ground state as mostly used in the literature. This will be based on choosing the harmonic oscillators as system and using the path integral technique as tools to deal with our issues. Our analysis will involve the temperature as an important parameter that will play a crucial role in discussing different properties of our system. To prove the validity of our approach, we will show that our results will allow to recover the standard case describing the quantum system in the ground state at absolute zero temperature, which have been obtained in our previous work dealing with the entanglement in coupled harmonic oscillators studied using a unitary transformation [12].

More precisely, we investigate the entanglement of a system of two coupled harmonic oscillators [13] by adopting the path integral formalism. Indeed, we build the propagator corresponding to our system and therefore derive its density matrix. This will allow us to determine the wavefunction describing the state of our system and subsequently end up with the purity in terms of different physical parameters. We emphasize that such wavefunction is temperature dependent and associated to the whole energy spectrum including the ground-state, which offers an exact derivation of the purity for our system. We study interesting cases of the purity by considering different limiting cases related to the temperature, coupling parameter and type of particles. At low temperature, we recover our former result obtained in [12] and at high temperature the purity shows another behavior. Jointly to our findings, we present different numerical results and discuss their basic features.

The present paper is organized as follows. In section 2, we construct the density matrix of bipartite corresponding to a system of two coupled harmonic oscillators. By examining the two limiting cases of the temperature parameter β of the density matrix and from the well-known solution of the imaginary

time Schrödinger equation [14] we derive the β -dependent wavefunction, which verifies the low and high temperature limits as well as allows to obtain the reduced density matrix. This will be done by making use of unitary transformation together with the initial conditions as well as different changes. In section 3, from the obtained results we show how to derive the purity in general form and study interesting cases related to the strengths of different physical quantities defining our system. We conclude our results in the final section and give some perspectives.

2 Path integral and bipartite

We consider a system of two coupled harmonic oscillators of masses (m_1, m_2) parameterized by the planar coordinates (x_1, x_2) . This is governed by a Hamiltonian sum of free and interacting parts

$$\hat{H} = \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} + \frac{1}{2}C_1\hat{x}_1^2 + \frac{1}{2}C_2\hat{x}_2^2 + \frac{1}{2}C_3\hat{x}_1\hat{x}_2 \quad (1)$$

where C_1, C_2 and C_3 are constant parameters. It is clear that the decoupled harmonic oscillators are recovered by requiring $C_3 = 0$. In the next, we will adopt the path formalism to explicitly determine the wavefunction corresponding to the present system and later on study the purity to characterize the strengths of the entanglement. In doing so, we proceed by introducing the density matrix and particularly the reduced density matrix.

2.1 Density matrix

In the beginning let us establish the mathematical tool that will be used to attack our concern and deal with different issues. Indeed, in constructing the the path integral for the propagator corresponding to the Hamiltonian (1), according to [14, 15] we consider the energy shift

$$\hat{H} \longrightarrow \hat{H} - E_0\hat{\mathbb{I}} \quad (2)$$

to ensure that the wavefunction of the system converges to that of the ground state at low temperature. Now let us introduce the evolution operator

$$\hat{\mathbf{U}}(\beta) = \mathcal{T}_D \exp \left(- \int_0^\beta (\hat{H} - E_0\hat{\mathbb{I}}) d\tau \right) = e^{+\beta E_0} \mathcal{T}_D \exp \left(- \int_0^\beta \hat{H} d\tau \right) \quad (3)$$

which involves the important parameter of our theory that is the temperature T with $\beta = \frac{1}{k_B T}$ and the Boltzmann constant k_B , \mathcal{T}_D being chronological Dyson operator. The matrix elements of such propagator take the form

$$\rho^{AB}(x_{1b}, x_{2b}, x_{1a}, x_{2a}; \beta) = \langle x_{1b}, x_{2b} | \hat{\mathbf{U}}(\beta) | x_{1a}, x_{2a} \rangle \quad (4)$$

where A and B are two subregions forming our system, with $|x_{1a}, x_{2a}\rangle$ and $|x_{1b}, x_{2b}\rangle$ are the initial and final states, respectively. In the forthcoming analysis, we consider the shorthand notation $\rho^{AB}(x_{1b}, x_{2b}, x_{1a}, x_{2a}; \beta) = \rho^{AB}(b, a; \beta)$. To write this latter in terms of the path integral, one can divide the temperature parameter β into $N + 1$ intervals of length $\epsilon = \frac{\beta}{N+1}$, use the Trotter formula and insert the completeness relation

$$\int |x_1, x_2\rangle \langle x_1, x_2| dx_1 dx_2 = \mathbb{I} \quad (5)$$

to get the continuous form

$$\rho^{AB}(b, a; \beta) = e^{+\beta E_0} \times \int Dx_1 Dx_2 \exp \left\{ - \int_0^\beta \left(\frac{m_1}{2} \dot{x}_1^2 + \frac{m_2}{2} \dot{x}_2^2 + \frac{1}{2} C_1 x_1^2 + \frac{1}{2} C_2 x_2^2 + \frac{1}{2} C_3 x_1 x_2 \right) d\tau \right\} \quad (6)$$

where the initial conditions $x_1(0) = x_{1a}, x_2(0) = x_{2a}, x_1(\beta) = x_{1b}, x_2(\beta) = x_{2b}$ will be taken into account in order to get the solutions.

To go further in developing the above density matrix, we proceed by introducing some relevant tools. Indeed, in the beginning we consider the unitary transformation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\mu} \cos \frac{\theta}{2} & \frac{1}{\mu} \sin \frac{\theta}{2} \\ -\mu \sin \frac{\theta}{2} & \mu \cos \frac{\theta}{2} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \quad (7)$$

where we have set the quantities

$$\tan \theta = \frac{C_3}{\mu^2 C_2 - \frac{C_1}{\mu^2}}, \quad \mu = \left(\frac{m_1}{m_2} \right)^{\frac{1}{4}}. \quad (8)$$

This transforms the density matrix (6) into the form

$$\rho^{AB}(b, a; \beta) = e^{+\beta E_0} \int DX_1 DX_2 \exp \left\{ - \int_0^\beta \left(\frac{m}{2} \dot{X}_1^2 + \frac{m}{2} \dot{X}_2^2 + \frac{1}{2} k e^{2\eta} X_1^2 + \frac{1}{2} k e^{-2\eta} X_2^2 \right) d\tau \right\} \quad (9)$$

and the involved parameters are given by

$$e^{\pm 2\eta} = \frac{\frac{C_1}{\mu^2} + \mu^2 C_2 \mp \sqrt{\left(\frac{C_1}{\mu^2} - \mu^2 C_2 \right)^2 + C_3^2}}{2k}, \quad k = \sqrt{C_1 C_2 - \frac{C_3^2}{4}}, \quad m = \sqrt{m_1 m_2}. \quad (10)$$

With the above transformation the system is now getting decoupled and therefore (9) is separable in terms of the new variables X_1 and X_2 . Consequently, we have

$$\rho^{AB}(X_{1b}, X_{2b}, X_{1a}, X_{2a}; \beta) = e^{+\beta E_0} \rho_1(X_{1b}, X_{1a}; \beta) \rho_2(X_{2b}, X_{2a}; \beta) \quad (11)$$

where the two parts read as

$$\rho_1 = \left(\frac{m\omega e^\eta}{2\pi\hbar \sinh(\hbar\omega\beta e^\eta)} \right)^{\frac{1}{2}} \exp \left[\frac{-m\omega e^\eta}{2\hbar \sinh(\hbar\omega\beta e^\eta)} \left((X_{1b}^2 + X_{1a}^2) \cosh(\hbar\omega\beta e^\eta) - 2X_{1b}X_{1a} \right) \right] \quad (12)$$

$$\rho_2 = \left(\frac{m\omega e^{-\eta}}{2\pi\hbar \sinh(\hbar\omega\beta e^{-\eta})} \right)^{\frac{1}{2}} \exp \left[\frac{-m\omega e^{-\eta}}{2\hbar \sinh(\hbar\omega\beta e^{-\eta})} \left((X_{2b}^2 + X_{2a}^2) \cosh(\hbar\omega\beta e^{-\eta}) - 2X_{2b}X_{2a} \right) \right] \quad (13)$$

with the frequency $\omega = \sqrt{\frac{k}{m}}$. Note that the main difference between both parts is the sign of the coupling parameter η , which will play a crucial role in the forthcoming analysis. Using the transformation (7) to map the density matrix of the composite system in terms of the original variables (x_1, x_2) as

$$\rho^{AB}(b, a; \beta) = \frac{m\omega}{2\pi\hbar} e^{+\beta E_0} \left(\frac{1}{\sinh(\hbar\omega\beta e^\eta) \sinh(\hbar\omega\beta e^{-\eta})} \right)^{\frac{1}{2}} \exp \left\{ -ax_{1b}^2 - bx_{2b}^2 - ax_{1a}^2 - bx_{2a}^2 \right\} \times \exp \left\{ 2cx_{1b}x_{2b} + 2cx_{1a}x_{2a} + 2dx_{1b}x_{1a} + 2fx_{2b}x_{2a} - 2gx_{1b}x_{2a} - 2gx_{1a}x_{2b} \right\} \quad (14)$$

where different quantities are

$$a = \mu^2 \frac{m\omega}{2\hbar} \left[e^\eta \coth(\hbar\omega\beta e^\eta) \cos^2 \frac{\theta}{2} + e^{-\eta} \coth(\hbar\omega\beta e^{-\eta}) \sin^2 \frac{\theta}{2} \right] \quad (15)$$

$$b = \frac{m\omega}{\mu^2 2\hbar} \left[e^\eta \coth(\hbar\omega\beta e^\eta) \sin^2 \frac{\theta}{2} + e^{-\eta} \coth(\hbar\omega\beta e^{-\eta}) \cos^2 \frac{\theta}{2} \right] \quad (16)$$

$$c = \frac{m\omega}{2\hbar} \left(e^\eta \coth(\hbar\omega\beta e^\eta) - e^{-\eta} \coth(\hbar\omega\beta e^{-\eta}) \right) \cos \frac{\theta}{2} \sin \frac{\theta}{2} \quad (17)$$

$$d = \frac{\mu^2 m\omega}{2\hbar} \left[\frac{e^\eta}{\sinh(\hbar\omega\beta e^\eta)} \cos^2 \frac{\theta}{2} + \frac{e^{-\eta}}{\sinh(\hbar\omega\beta e^{-\eta})} \sin^2 \frac{\theta}{2} \right] \quad (18)$$

$$f = \frac{m\omega}{\mu^2 2\hbar} \left[\frac{e^\eta}{\sinh(\hbar\omega\beta e^\eta)} \sin^2 \frac{\theta}{2} + \frac{e^{-\eta}}{\sinh(\hbar\omega\beta e^{-\eta})} \cos^2 \frac{\theta}{2} \right] \quad (19)$$

$$g = \frac{m\omega}{2\hbar} \left(\frac{e^\eta}{\sinh(\hbar\omega\beta e^\eta)} - \frac{e^{-\eta}}{\sinh(\hbar\omega\beta e^{-\eta})} \right) \cos \frac{\theta}{2} \sin \frac{\theta}{2}. \quad (20)$$

In the next, we will show how (14) can be used to deal with different issues related to our system behavior. More precisely, it will help in getting the corresponding wavefunction describing the whole energy spectrum of our system and later on derive its purity. This will be used to analyze the degree of the entanglement and obtain different results.

2.2 Reduced density matrix

To analyze the entanglement of our system, we need first to determine the reduced density matrix and later evaluate the purity. In doing so, in early stage we have to derive the wavefunction corresponding to our system described by the Hamiltonian (1), which has to satisfy the imaginary time Schrödinger equation and gives rise two wavefunctions at low and high temperature limits. This will be used to explicitly determine without approximation the corresponding purity in terms of different physical parameters. To get such suitable wavefunction, we start by making the variable substitution $(x_{1a}, x_{2a}) = (x_{1b}, x_{2b}) = (x_1, x_2)$ into (14) and taking only the diagonal elements of the density matrix, then we have the probability density

$$P_\beta(x_1, x_2) = \text{diag}(\rho^{AB}(b, a; \beta)) \quad (21)$$

which can be evaluated to get the form

$$P_\beta(x_1, x_2) = \frac{m\omega e^{+\beta E_0}}{2\pi\hbar \sqrt{\sinh(\hbar\omega\beta e^\eta) \sinh(\hbar\omega\beta e^{-\eta})}} e^{-\tilde{a}(\beta)x_1^2 - \tilde{b}(\beta)x_2^2 + 2\tilde{c}(\beta)x_1x_2} \quad (22)$$

where we have introduced the shorthand notations

$$\tilde{a}(\beta) = 2(a - d) = \mu^2 \frac{m\omega}{\hbar} \left[e^\eta \tanh\left(\frac{\hbar\omega}{2}\beta e^\eta\right) \cos^2 \frac{\theta}{2} + e^{-\eta} \tanh\left(\frac{\hbar\omega}{2}\beta e^{-\eta}\right) \sin^2 \frac{\theta}{2} \right] \quad (23)$$

$$\tilde{b}(\beta) = 2(b - f) = \frac{m\omega}{\mu^2 \hbar} \left[e^\eta \tanh\left(\frac{\hbar\omega}{2}\beta e^\eta\right) \sin^2 \frac{\theta}{2} + e^{-\eta} \tanh\left(\frac{\hbar\omega}{2}\beta e^{-\eta}\right) \cos^2 \frac{\theta}{2} \right] \quad (24)$$

$$\tilde{c}(\beta) = 2(c - g) = \frac{m\omega}{\hbar} \left[e^\eta \tanh\left(\frac{\hbar\omega}{2}\beta e^\eta\right) - e^{-\eta} \tanh\left(\frac{\hbar\omega}{2}\beta e^{-\eta}\right) \right] \cos \frac{\theta}{2} \sin \frac{\theta}{2}. \quad (25)$$

Note that, (22) looks like the probability density corresponding to the ground-state of two-dimensional harmonic oscillator but involving an interesting term that is the last one with $\tilde{c}(\beta)$ parameter. Clearly, if $\tilde{c}(\beta) = 0$ we end up with the solution of decoupled harmonic oscillators.

At this level, let us check the validity of the probability density (22) by examining two limiting cases related to the temperature parameter β . In the classical limit of high temperature, the probability distribution takes the form

$$\begin{aligned} P_{\beta_0}(x_1, x_2) &= \lim_{\beta \rightarrow 0} P_{\beta}(x_1, x_2) \sim \rho_{cl}(x_1, x_2; \beta_0) \\ &= \frac{m e^{+\beta_0 E_0}}{2\pi \hbar^2 \beta_0} \exp[-\beta_0 V(x_1, x_2)] \end{aligned} \quad (26)$$

where $V(x_1, x_2)$ is the potential energy

$$V(x_1, x_2) = \frac{1}{2} C_1 x_1^2 + \frac{1}{2} C_2 x_2^2 + \frac{1}{2} C_3 x_1 x_2 \quad (27)$$

and here we have involved the high temperature value β_0 . It is clearly seen that $P_{\beta_0}(x_1, x_2)$ is nothing but the Boltzmann probability distribution depending only on the potential energy $V(x_1, x_2)$. Such result is in agreement with the comments of the Kleinert's book [16]. However, in the opposite limit of low temperature we end up with the the probability distribution associated to the ground-state

$$\begin{aligned} \lim_{\beta \rightarrow \infty} P_{\beta}(x_1, x_2) &\sim \frac{m\omega}{\hbar\pi} e^{+\beta(E_0 - \hbar\omega \cosh \eta)} e^{-\frac{m\omega}{2\hbar} e^{\eta} \left(\mu \cos \frac{\theta}{2} x_1 - \frac{1}{\mu} \sin \frac{\theta}{2} x_2\right)^2 - \frac{m\omega}{2\hbar} e^{-\eta} \left(\mu \sin \frac{\theta}{2} x_1 + \frac{1}{\mu} \cos \frac{\theta}{2} x_2\right)^2} \\ &= |\psi_0(x_1, x_2)|^2 \end{aligned} \quad (28)$$

and it is interesting to stress that such limit of temperature imposes the ground-state energy should be of the form

$$E_0 = \hbar\omega \cosh \eta \quad (29)$$

giving rise the ground-state wavefunction

$$\psi_0(x_1, x_2) = \sqrt{\frac{m\omega}{\hbar\pi}} e^{-\frac{m\omega}{2\hbar} e^{\eta} \left(\mu \cos \frac{\theta}{2} x_1 - \frac{1}{\mu} \sin \frac{\theta}{2} x_2\right)^2 - \frac{m\omega}{2\hbar} e^{-\eta} \left(\mu \sin \frac{\theta}{2} x_1 + \frac{1}{\mu} \cos \frac{\theta}{2} x_2\right)^2} \quad (30)$$

which are in agreement with those obtained in or previous work [12] just by fixing the quantum numbers $n_1 = n_2 = 0$.

Generally for any temperature parameter β , the wavefunction describing our system can be determined by integrating over the initial variables as has been done in [14]. Thus, in our case we have to write the solution of the imaginary time Schrödinger equation as

$$\psi(x_1, x_2; \beta) = \int \rho^{AB} \left(b, a; \beta - \frac{\varepsilon}{2}\right) \psi \left(x_{1a}, x_{2a}; \frac{\varepsilon}{2}\right) dx_{1a} dx_{2a} \quad (31)$$

where the density matrix of the system verifies the condition

$$\lim_{\beta \rightarrow \frac{\varepsilon}{2}} \rho^{AB} \left(b, a; \beta - \frac{\varepsilon}{2}\right) = \delta(x_1 - x_{1a}) \delta(x_2 - x_{2a}). \quad (32)$$

Note that in order to eliminate $\frac{\varepsilon}{2}$ introduced in the initial wavefunction we have written the propagator $\rho^{AB} \left(b, a; \beta - \frac{\varepsilon}{2}\right)$ with the prescription $\frac{\varepsilon}{2}$ and in this case (31) is nothing but the exact convolution product. To go further it is clear that one has to fix the form of the initial wavefunction $\psi \left(x_{1a}, x_{2a}; \frac{\varepsilon}{2}\right)$ and then from (26) we choose a non-normalized initial wavefunction as

$$\psi \left(x_{1a}, x_{2a}; \frac{\varepsilon}{2}\right) = \frac{\sqrt{\frac{m\omega}{4\pi\hbar}}}{\sqrt{\cosh(\hbar\omega e^{\eta} \frac{\varepsilon}{2}) \cosh(\hbar\omega e^{-\eta} \frac{\varepsilon}{2})}} e^{-\frac{1}{2} \tilde{a}(\varepsilon) x_1^2 - \frac{1}{2} \tilde{b}(\varepsilon) x_2^2 + \tilde{c}(\varepsilon) x_1 x_2} \quad (33)$$

where ε is a small value of the high temperature, which is introduced to insure the convergence of the probability density of the initial state. Clearly $\psi(x_{1a}, x_{2a}; \frac{\varepsilon}{2})$ does not verify the Schrödinger equation but its square absolute value $|\psi(x_{1a}, x_{2a}; \frac{\varepsilon}{2})|^2$ is proportional to the probability density (26). Now replacing (33) and integrating (31) to end up with the suitable wavefunction for the whole energy spectrum

$$\psi(x_1, x_2; \beta) = \frac{\sqrt{\frac{m\omega}{4\pi\hbar}}}{\sqrt{\cosh(\hbar\omega e^\eta \beta) \cosh(\hbar\omega e^{-\eta} \beta)}} e^{+\beta\hbar\omega \cosh \eta} e^{-\tilde{\alpha}x_1^2 - \tilde{\beta}x_2^2 + 2\tilde{\gamma}x_1x_2} \quad (34)$$

and the quantities take the form

$$\tilde{\alpha} = \mu^2 \frac{m\omega}{2\hbar} [e^\eta \tanh(\hbar\omega e^\eta \beta) \cos^2 \frac{\theta}{2} + e^{-\eta} \tanh(\hbar\omega e^{-\eta} \beta) \sin^2 \frac{\theta}{2}] \quad (35)$$

$$\tilde{\beta} = \frac{m\omega}{2\mu^2\hbar} [e^\eta \tanh(\hbar\omega e^\eta \beta) \sin^2 \frac{\theta}{2} + e^{-\eta} \tanh(\hbar\omega e^{-\eta} \beta) \cos^2 \frac{\theta}{2}] \quad (36)$$

$$\tilde{\gamma} = \frac{m\omega}{2\hbar} [e^\eta \tanh(\hbar\omega e^\eta \beta) - e^{-\eta} \tanh(\hbar\omega e^{-\eta} \beta)] \cos \frac{\theta}{2} \sin \frac{\theta}{2}. \quad (37)$$

At this level, we have two interesting comments in order. Indeed, firstly it is interesting to stress that the obtained wavefunction is temperature dependent and satisfies the imaginary time Schrödinger equation

$$\left(\hat{H} - E_0 \hat{\mathbb{1}}\right) \psi(x_1, x_2; \beta) + \frac{\partial}{\partial \beta} \psi(x_1, x_2; \beta) = 0 \quad (38)$$

where the substitution $t \rightarrow -i\hbar\beta$ is taken into account. Secondly, in both limiting cases of temperature, $\psi(x_1, x_2; \beta)$ (34) converges to the initial wavefunction (33) at high temperature ($\beta \rightarrow \frac{\varepsilon}{2}$) as well as to the ground-state wavefunction (30) at low temperature.

Once the wavefunction corresponding to our system is obtained, we now return back to explicitly determine the reduced density matrix. Then based on the standard definition

$$\rho_{\text{red}}^A(x_1, x'_1; \beta) = \frac{\int \psi(x_1, x_2; \beta) \psi^*(x'_1, x_2; \beta) dx_2}{\int \psi(x_1, x_2; \beta) \psi^*(x_1, x_2; \beta) dx_1 dx_2} \quad (39)$$

we show that the one-particle reduced density matrix takes the form

$$\rho_{\text{red}}^A(x_1, x'_1; \beta) = A(\beta) \exp\left(-\frac{2\tilde{\alpha}\tilde{\beta} - \tilde{\gamma}^2}{2\tilde{\beta}} x_1^2 - \frac{2\tilde{\alpha}\tilde{\beta} - \tilde{\gamma}^2}{2\tilde{\beta}} x_1'^2 + \frac{\tilde{\gamma}^2}{\tilde{\beta}} x_1 x_1'\right) \quad (40)$$

where the normalization factor $A(\beta)$ verifies the von Neumann's normalization $\text{Tr}_A(\rho_{\text{red}}^A(x_1, x'_1; \beta)) = 1$ and is given by

$$A(\beta) = \sqrt{2 \frac{\tilde{\alpha}\tilde{\beta} - \tilde{\gamma}^2}{\pi\tilde{\beta}}}. \quad (41)$$

We close this part by noting that in the present analysis we did not require a normalized wavefunction to determine the reduced matrix density because such type of function does not satisfy the imaginary time Schrödinger equation (38).

3 Purity function

Having obtained the necessary materials, we move to the second step and derive the general form of the purity function corresponding to our system. Later on, we will study interesting limiting cases of such purity in terms of the strengths of different physical parameters. To underline our system behavior, we will numerically illustrate the obtained results and give some discussions.

3.1 General case

In general, the degree of information about the preparation of a quantum state can be characterized by the associated purity. In other term, to quantify the mixedness of a state ρ one can use the purity [17,18]

$$P = \text{Tr}_A (\rho)^2 \quad (42)$$

where for d -dimensional systems P ranges from $1/d$ for completely mixed states to 1 for pure states. It is closely related to the linear entropy measure of mixedness [19]

$$S = \frac{d}{d-1} (1 - P) \quad (43)$$

which ranges from 0 (for a pure state) to 1 (for a maximally mixed state). The linear entropy is generally a simpler quantity to calculate than the von Neumann entropy as there is no need for diagonalization.

Before determining the purity, let us first stress that the reduced density matrix ρ_{red}^B corresponding to subregion B has the same form as in (40), which can also be obtained by integrating (21) over the variable x_1 . Therefore, for both subregions A and B we have the same purity, which will be denoted by $P^A = P^B = P$. It is defined as trace over square of the reduced density matrix, such as

$$P = \text{Tr}_A (\rho_{\text{red}}^A(x_1, x'_1; \beta))^2 \quad (44)$$

or equivalently by using (40) together with (41) we obtain

$$\begin{aligned} P &= \int \rho_{\text{red}}^A(x_1, x'_1; \beta) \rho_{\text{red}}^A(x'_1, x_1; \beta) dx_1 dx'_1 \\ &= A^2(\beta) \int e^{-\frac{2\tilde{\alpha}\tilde{\beta}-\tilde{\gamma}^2}{\tilde{\beta}}x_1^2 - 2\frac{2\tilde{\alpha}\tilde{\beta}-\tilde{\gamma}^2}{2\tilde{\beta}}x_1'^2 + 2\frac{\tilde{\gamma}^2}{\tilde{\beta}}x_1x_1'} dx_1 dx'_1. \end{aligned} \quad (45)$$

After integral performance, we end up with the compact form

$$P = \sqrt{\frac{\tilde{\alpha}\tilde{\beta} - \tilde{\gamma}^2}{\tilde{\alpha}\tilde{\beta}}} \quad (46)$$

and also we can replace different quantities and make straightforward calculation to get the explicit form

$$P = \sqrt{\frac{\tanh(\hbar\omega\beta e^\eta) \tanh(\hbar\omega\beta e^{-\eta})}{(e^\eta \tanh(\hbar\omega\beta e^\eta) \sin^2 \frac{\theta}{2} + e^{-\eta} \tanh(\hbar\omega\beta e^{-\eta}) \cos^2 \frac{\theta}{2}) (e^\eta \tanh(\hbar\omega\beta e^\eta) \cos^2 \frac{\theta}{2} + e^{-\eta} \tanh(\hbar\omega\beta e^{-\eta}) \sin^2 \frac{\theta}{2})}} \quad (47)$$

which is showing a strong dependence on the external parameters η , θ and temperature β . One more thing, P is actually the product of two quantities and they are differentiating by the η sign of the numerator and the geometric functions in the denominator. Moreover, we notice that the obtained purity is actually based on exact calculation without use of approximation and it is corresponding to the whole energy spectrum. Consequently to underline our system behavior versus different configurations of the physical parameters we numerically investigate the basic features of the purity (47). For the numerical use in the next, it is convenient to consider the rescaling $\hbar\omega\beta \rightarrow \beta$ giving rise a dimensionless parameter.

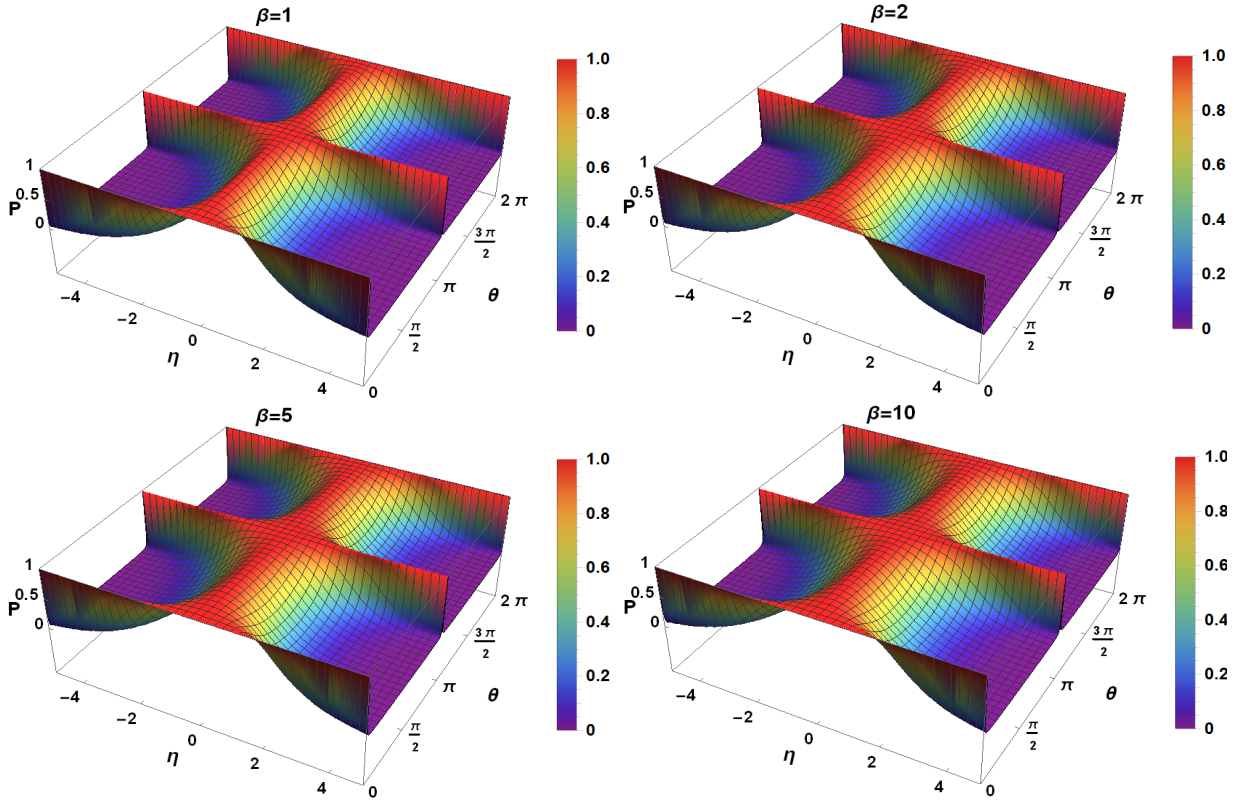


Figure 1 – Purity versus the coupling parameter η and the mixing angle θ for four fixed value of the temperature $\beta = 1, 2, 5, 10$.

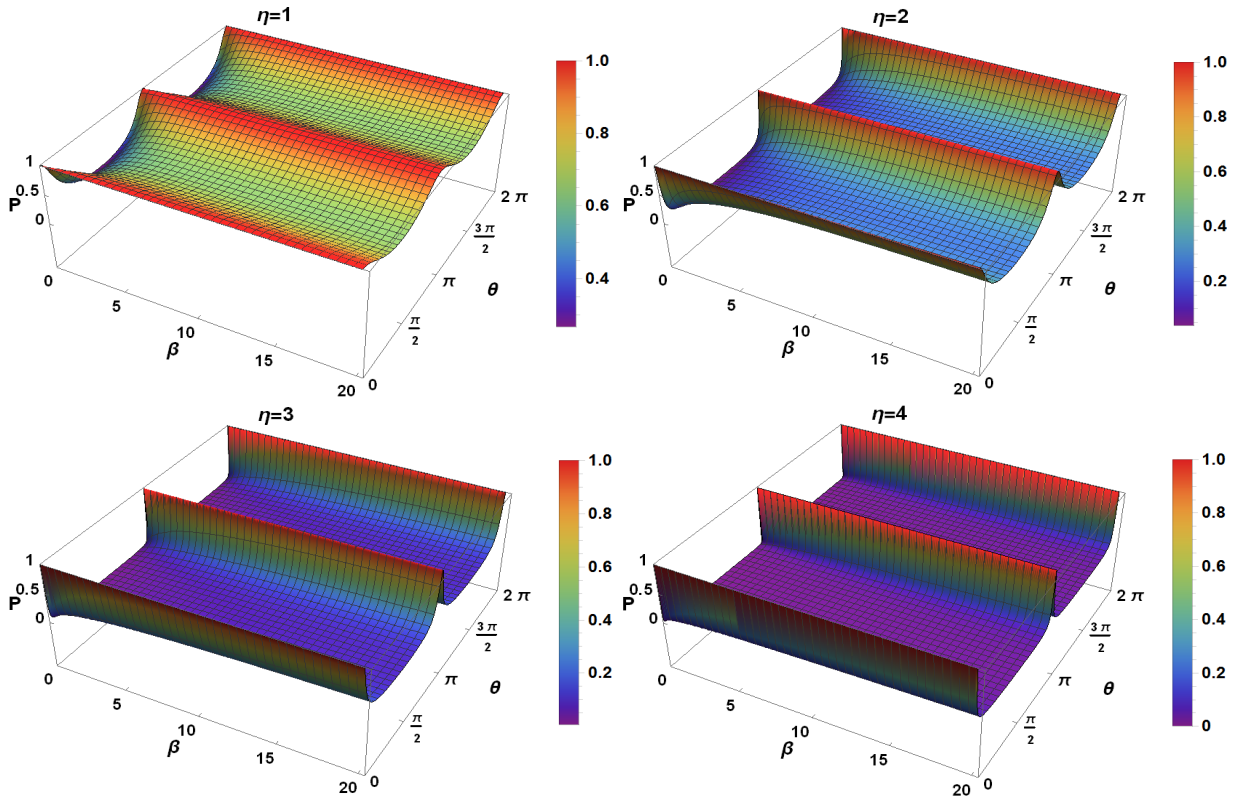


Figure 2 – Purity versus the temperature β and the mixing angle θ for four fixed value of the coupling parameter $\eta = 1, 2, 3, 4$.

In Figure 1, we plot the purity versus the coupling parameter η and the mixing angle $\theta \in [0, 2\pi]$ for four fixed value of the temperature $\beta = 1, 2, 5, 10$. It is clear that the purity, as a function of η and θ is symmetric with respect to the case $\eta = 0$ and the decoupling case $\theta = \pi$ and it lies in the interval $[0, 1]$. It is maximal for $\eta = 0$ as well for $\theta = \pi$, which actually shows that the system is disentangled. After that it decreases rapidly to reach zero and this indicates that the entanglement is maximal. More importantly, the purity becomes constant whenever θ takes the value zero or 2π . This behavior of the purity, tells us that one can easily play with two parameters to control the degree of entanglement in our system. As one can see from the four plots that the purity increases as long as temperature is decreased. It means that the temperature plays a great role in destroying the system entanglement.

Figure 2 shows the purity versus the temperature β and the rotating angle θ for four fixed value of the coupling parameter $\eta = 1, 2, 3, 4$. In such situation, we observe that the purity presents another behavior rather than that observed in Figure 1. Indeed, it converges rapidly to the minimal and maximal values as long as the coupling parameter is increased. More precisely from plot with $\eta = 1$ to that with $\eta = 4$ there is a radical changes from highly entangled system to a minimal entanglement.

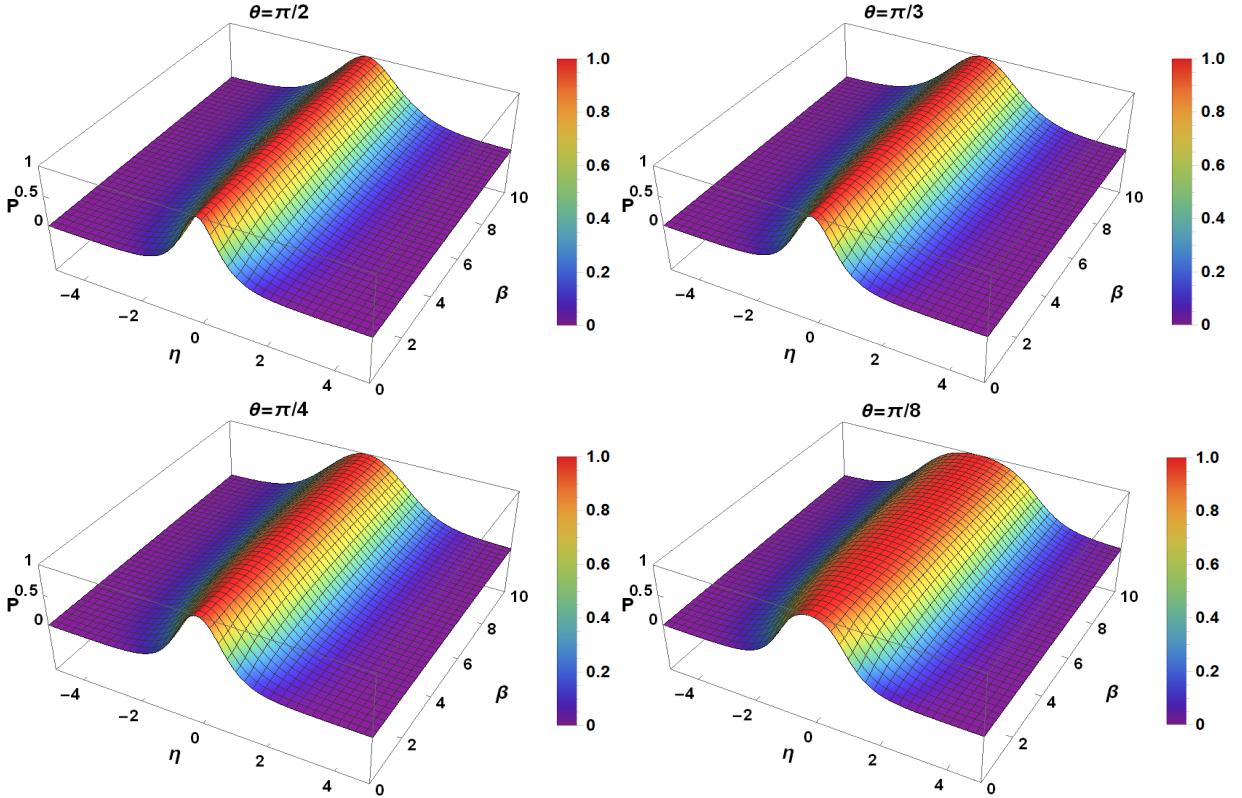


Figure 3 – Purity versus the coupling parameter η and the temperature β for four fixed values of the mixing angle $\theta = \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{8}$.

To accomplish such numerical analysis, we inspect the last case by plotting the purity versus the coupling parameter η and the temperature β for four fixed values of the rotating angle $\theta = \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{8}$ in Figure 3. It is clearly seen that the purity is increasing slowly from $\theta = \frac{\pi}{2}$ to $\theta = \frac{\pi}{8}$, which is showing some minima and maxima. However the purity takes a maximal value when the coupling parameter η is null and also it has a symmetrical behavior.

3.2 Special cases

It is interesting to study some limiting cases in order to characterize our system behavior. Since our theory involves three free parameters, one can examine different situations according to some chosen configurations to simplify the purity function and therefore extract more information about the entanglement of our system. These concern the temperature limits, coupling limits and finally a system composed of identical particles.

3.2.1 Temperature limits

We start our study by looking at the asymptotic behavior of our system with respect to the temperature parameter β limits. Indeed, to analyze the low temperature case we take the limit $\beta \rightarrow \infty$ in the general form of the purity (47). Doing this process to get the interesting result

$$P(\eta, \theta; \beta \rightarrow \infty) = \frac{1}{\left| \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right| \sqrt{\left(2 \cosh(2\eta) + \tan^2 \frac{\theta}{2} + \cot^2 \frac{\theta}{2} \right)}} \quad (48)$$

which ensures that our system is actually in the ground-state and therefore it is exactly the purity $P_{0,0}(\eta, \theta)$ corresponding to the ground-state ($n_1 = 0, n_2 = 0$) obtained in our previous work in dealing with the entanglement of two coupled harmonic oscillators studied using unitary transformation [12]. This result confirms that considering the environment temperature as a reservoir acting on our system is a good candidate to describe its degree of entanglement.

Now let us consider the high temperature regime and analyze the degree of the entanglement of our system in such situation. This can be worked out by taking the limit $\beta \rightarrow \frac{\varepsilon}{2}$ in (47) to end up with the purity form

$$P\left(\eta, \theta; \beta \rightarrow \frac{\varepsilon}{2}\right) = \sqrt{\frac{1}{\left(e^{2\eta} \sin^2 \frac{\theta}{2} + e^{-2\eta} \cos^2 \frac{\theta}{2} \right) \left(e^{2\eta} \cos^2 \frac{\theta}{2} + e^{-2\eta} \sin^2 \frac{\theta}{2} \right)}}. \quad (49)$$

To make comparison with respect to the previous limiting case, we can use straightforward algebra to rearrange such relation as

$$\begin{aligned} P\left(\eta, \theta; \beta \rightarrow \frac{\varepsilon}{2}\right) &= \frac{1}{\left| \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right| \sqrt{2 \cosh(4\eta) + \tan^2 \frac{\theta}{2} + \cot^2 \frac{\theta}{2}}} \\ &= \sqrt{\frac{C_1 C_2 - \frac{C_3^2}{4}}{C_1 C_2}}. \end{aligned} \quad (50)$$

It is clear that now we can establish a link between both of temperature regimes and therefore write the following relation

$$P\left(\eta, \theta; \beta \rightarrow \frac{\varepsilon}{2}\right) = P(2\eta, \theta; \beta \rightarrow \infty) \quad (51)$$

which tells us that it is matter of controlling the values taken by the coupling parameter η to go from a regime to another.

Figure 4 shows the high temperature regime for the purity as function of the coupling parameter η and the mixing angle θ . We observe that P increases slowly and at some interval of η with $\theta \in [0, \pi]$, it reaches the maximum where the system becomes highly disentangled and the same behavior happened for $\theta \in [\pi, 2\pi]$. Also P shows a symmetrical behavior for two values $\eta = 0$ and $\theta = \pi$, which are corresponding to the maximal value of P .

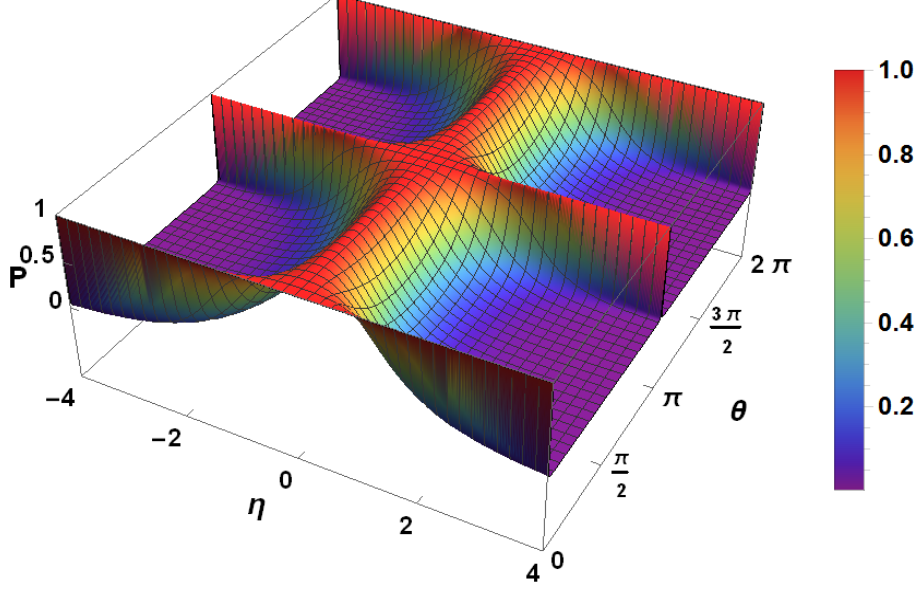


Figure 4 – Purity versus the coupling parameter η and the mixing angle θ for high temperature limit.

3.2.2 Coupling limits

Two situations will be analyzed with respect to the strength of the coupling parameter η , which will allow us to see how much our system is entangled. We start with the weak coupling that is characterized by taking the limit $C_3 \rightarrow 0$ where the angle $\theta \rightarrow \theta_w$ and the coupling $\eta \rightarrow \eta_w$. In this case, (8) and (10) reduce to the following

$$e^{2\eta_w} = \frac{1}{\mu^2} \sqrt{\frac{C_1}{C_2}}, \quad \theta_w = 0 \quad (52)$$

which can be implemented into (47) to get the maximal value of the purity

$$P(\eta_w, \theta_w; \beta) = 1 \quad (53)$$

and therefore it is showing that our system is completely separable and consequently there is no entangled states.

Now we consider the strong coupling limit and derive the corresponding purity. In doing so, we notice that if the limit $C_3 \rightarrow 2\sqrt{C_1 C_2}$ is required then one could obtain

$$\tan \theta_s \rightarrow \frac{2\sqrt{C_1 C_2}}{\mu^2 C_2 - \frac{C_1}{\mu^2}} \rightarrow 0 \quad (54)$$

$$\eta \rightarrow \eta_s = +\infty, \quad k \rightarrow 0^+. \quad (55)$$

Combining all to end up with the results

$$k e^{2\eta_s} \rightarrow \frac{C_1}{\mu^2} + \mu^2 C_2, \quad \theta_s = \tan^{-1} \left(\frac{2\sqrt{C_1 C_2}}{\mu^2 C_2 - \frac{C_1}{\mu^2}} \right) \quad (56)$$

and thus the purity (47) reduces to the following quantity

$$P^A(\eta_s, \theta_s; \beta) \rightarrow 0 \quad (57)$$

which is telling us that our system is maximally entangled. This summarize that there are two extremely values of the purity those could be reached as long as the coupling parameter takes small or large values.

3.2.3 Identical particles

The last situation is related to the nature of our system, which is equivalent to require that both of harmonic oscillators have the same mass $m_1 = m_2$ and frequency $C_1 = C_2$. Thus from (8) and (10), we end up with the constraint $\theta \rightarrow \frac{\pi}{2}$ and $\eta \rightarrow \eta_{id}$ with

$$e^{2\eta_{id}} = \sqrt{\frac{C_1 + \frac{C_3}{2}}{C_1 - \frac{C_3}{2}}} \quad (58)$$

and replacing in the purity (47) to obtain

$$P\left(\eta_{id}, \theta = \frac{\pi}{2}; \beta\right) = \frac{2\sqrt{\tanh\left(\hbar\sqrt{\frac{C_1 + \frac{C_3}{2}}{m_1}}\beta\right)\tanh\left(\hbar\sqrt{\frac{C_1 - \frac{C_3}{2}}{m_1}}\beta\right)}}{\left(\frac{C_1 + \frac{C_3}{2}}{C_1 - \frac{C_3}{2}}\right)^{\frac{1}{4}}\tanh\left(\hbar\sqrt{\frac{C_1 + \frac{C_3}{2}}{m_1}}\beta\right) + \left(\frac{C_1 - \frac{C_3}{2}}{C_1 + \frac{C_3}{2}}\right)^{\frac{1}{4}}\tanh\left(\hbar\sqrt{\frac{C_1 - \frac{C_3}{2}}{m_1}}\beta\right)} \quad (59)$$

which is depending on the temperature. We notice that there is a special value of the coupling parameter reducing the above expression to a simplified form. Indeed, by requiring $C_3 = \frac{30}{17}C_1$, we obtain $\eta_{id} = \ln 2$ and therefore the purity becomes

$$P\left(\eta_{id} = \ln 2, \theta = \frac{\pi}{2}; \beta\right) = \frac{2\sqrt{\tanh\left(\hbar\sqrt{\frac{32C_1}{17m_1}}\beta\right)\tanh\left(\hbar\sqrt{\frac{2C_1}{17m_1}}\beta\right)}}{2\tanh\left(\hbar\sqrt{\frac{32C_1}{17m_1}}\beta\right) + \frac{1}{2}\tanh\left(\hbar\sqrt{\frac{2C_1}{17m_1}}\beta\right)} \quad (60)$$

which converges to a maximal value at low temperature

$$\lim_{\beta \rightarrow \infty} P\left(\eta_{id} = \ln 2, \theta = \frac{\pi}{2}; \beta\right) = \frac{4}{5}. \quad (61)$$

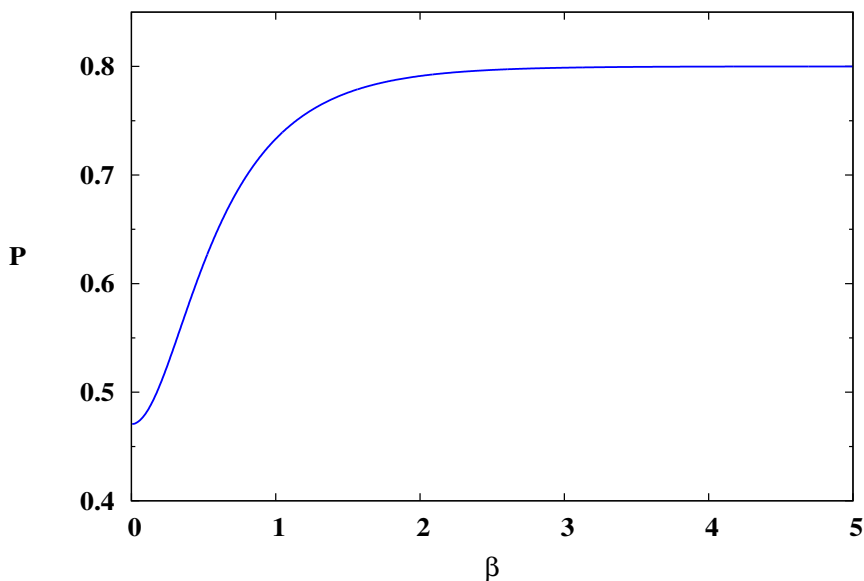


Figure 5 – Purity versus the temperature for the special values of $\eta_{id} = \ln 2$ and $\theta = \frac{\pi}{2}$.

We plot the reduced purity in terms of the temperature β as shown in Figure 5. One observes that at high temperature the purity is fixed to a minimal value, as long as the temperature is decreased the purity rapidly increases to reach a maximal value 0.8 at very low temperature.

4 Conclusion

We have analyzed the entanglement of a system of two coupled harmonic oscillators by using the path integral mechanism. For this, we have involved a global propagator based on temperature evolution of our system. Considering a unitary transformation we were able to explicitly obtain the reduced density matrix and therefore the wavefunction describing the whole spectrum of our system. Later on, we have shown that at high temperature regime one could obtain the classical distribution that is the Boltzmann probability and also at low temperature regime the system is in the ground state as should be.

Subsequently, we have used the reduced matrix density to discuss the entanglement of our system. For this, in the first stage we have calculated the corresponding purity function, which has been obtained in terms of different physical parameters. These allowed us to present different numerical results according to choice of some configurations of such parameters and therefore extract information about the system behavior. In fact in different occasions, we have obtained the degree of the entanglement varying between maxima and minima values.

We close by mentioning some extension of the present work. Indeed, one could ask about the corresponding entropies of Rényi and von Neumann. Also, we can ask about the inseparability and the uncertainty relations by defining the associated Podolsky-Einstein-Rosen operator [20]. These issues and related matters are actually under consideration.

Acknowledgment

We thank Youness Zahidi for his numerical help. The authors acknowledge the financial support from King Faisal University.

References

- [1] E. Schrödinger, *Naturwissenschaften* **23** (1935) 807.
- [2] A. Einstein, B. Podolsky and N. Rosen, *Phys. Rev.* **47** (1935) 777.
- [3] J.S. Bell, *"Speakable and Unsayable in Quantum Mechanics"* (Cambridge University Press, 1987).
- [4] Charles. H. Bennett and Peter W. Shor, *IEEE Transactions on Information Theory* **44** (1998) 2724.
- [5] C.H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres and W.K. Wootters, *Phys. Rev. Lett.* **70** (1993) 1895.

- [6] C.H. Bennett and S.J. Wiesner, Phys. Rev. Lett. **69** (1992) 2881.
- [7] A.K. Ekert, Phys. Rev. Lett. **67** (1991) 661.
- [8] M. Murao, D. Jonathan, M.B. Plenio and V. Vedral, Phys. Rev. A **59** (1999) 156.
- [9] C.A. Fuchs, Phys. Rev. Lett. **79** (1997) 1162.
- [10] R. Rausschendorf and H. Briegel, Quantum computing via measurements only, [quant-ph/0010033](#).
- [11] D. Gottesman and I. Chuang, Nature **402** (1999) 390.
- [12] A. Jellal, F. Madouri and A. Merdaci, J. Stat. Mech. (2011) P09015.
- [13] A. Jellal, E.H. El Kinani and M. Schreiber, Int. J. Mod. Phys. A **20** (2005) 1515.
- [14] I. Kosztin, B. Faber and K. Schulten, Am. J. Phys. **64** (1996) 633.
- [15] M. Rossi, M. Nava, L. Reatto and D. E. Galli, J. Chem. Phys **131** (2009) 154108.
- [16] H. Kleinert, " *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets* " (World Scientific, Singapore 2009).
- [17] G. Adesso, A. Serafini and F. Illuminati, Open Syst. Inf. Dyn. **12** (2005) 189.
- [18] D. McHugh, M. Ziman and V. Buzek, Phys. Rev. A. **74** (2006) 042303.
- [19] T-C Wei, K. Nemoto, P. M. Goldbart, P. G. Kwiat, W. J. Munro and F. Verstraete, Phys. Rev. A **67** (2003) 022110.
- [20] Lu-Ming Duan, G. Giedke, J. I. Cirac and P. Zoller, Phys. Rev. Lett. **84** (2000) 2722.