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Subclasses of Spiralike Functions Involving Convoluted Differential Operator

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Abstract

A subclasses of spiralike functions defined by a convoluted operator are introduced. Convolution properties, necessary and sufficient condition, and coefficient bounds of these subclasses are obtained.

Keywords: *Analytic function, spiralike function, differential operator, convolution, subordination.*

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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and normalised by the conditions $f(0) = 0$ and $f'(0) = 1$.

A subordination between two analytic functions f and g is written as $f \prec g$. Conceptually, the analytic function f is subordinate to g if the image under g contains the image under f. Technically, the analytic function f is subordinate to g if there exists a Schwarz function w with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{U}$; such that

$$f(z) = g(w(z)).$$

Besides, if the function g is univalent in $\mathbb{U}$, then the following equivalence holds:

$$f(z) \prec g(z) \quad \text{if and only if} \quad f(0) = g(0)$$

and

$$f(\mathbb{U}) \subset g(\mathbb{U}).$$

A convolution between two analytic functions f and g is written as $f * g$. Conceptually, convolution function of f and g expresses how the shape of one is modified by the other. Technically, convolution function of

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad \text{and} \qquad g(z) = z + \sum_{k=2}^{\infty} b_k z^k$$

is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

If we consider convolution as operation, then the right half-plane mapping $\ell(z) = z/(1-z)$, acts as the identity of convolution (cf [14], p290); that is, if f is an analytic function, then

$$\frac{z}{1-z} * f(z) = f(z) * \frac{z}{1-z} = f(z).$$

Also, Koebe function $\kappa(z) = z/(1-z)^2$ and other certain functions act as differential operators. We can see that clearly from the following examples:

$$\frac{z}{(1-z)^2} * f(z) = zf'(z), \tag{2}$$

$$\frac{z^2}{(1-z)^2} * f(z) = zf'(z) - f(z), \tag{3}$$

$$\frac{z}{(1-z)^3} * f(z) = \frac{z}{2}\left(zf(z)\right)'', \tag{4}$$

and

$$\frac{z^2}{(1-z)^3} * f(z) = \frac{z}{2}\left(zf(z)\right)'' - zf'(z). \tag{5}$$

A function in the class $\mathcal{A}$ is said to be μ-spiralike, $S_p(\mu)$, if and only if

$$\Re\left\{e^{i\mu}\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in \mathbb{U},$$

where μ is real number with $|\mu| < \pi/2$. This class was introduced and shown to be univalent by Špaček [22]. Sharp coefficient bounds of $S_p(\mu)$ was obtained by Zamorski [23]. Note here, $S_p(0) \equiv S^*$, which is the well-known class of starlike functions, and $S_p(\pi/2)$ is nothing but the identity function $f(z) = z$ which is out of our consideration.

Libera [17], introduced the class of μ-spiralike functions of order λ, $0 \leq \lambda \leq 1$, denoted by $S_p(\mu, \lambda)$ to be the set of functions of the form (1) that satisfy

$$\Re\left\{e^{i\mu}\frac{zf'(z)}{f(z)}\right\} > \lambda\cos\mu, \quad z \in \mathbb{U}.$$

Libera discussed a geometric interpretation of $S_p(\mu, \lambda)$ and used characterisation that f in the class μ-spiralike of order $\lambda(\cos\mu)^{-1}$ if and only if

$$e^{i\mu}\frac{zf'(z)}{f(z)} \prec \left\{\frac{1-(1-2\lambda)z}{1-z}\right\}(\cos\mu) + i\sin\mu, \qquad z \in \mathbb{U}. \tag{6}$$

to obtain sharp coefficient bounds for the class.

Furthermore in [17], coefficient bounds and radius of μ-spiralikeness were obtained.

Dashreth and Shukla [16], introduced a class of μ-spiralike functions denoted by $\mathcal{S}^\mu[A, B]$ which is the set of functions in $\mathcal{A}$ that satisfies

$$e^{i\mu}\frac{zf'(z)}{f(z)} \prec (\cos\mu)\left(\frac{1+Az}{1+Bz}\right) + i\sin\mu, \qquad z \in \mathbb{U}, \tag{7}$$

where μ, A, and B are real with $|\mu| < \frac{\pi}{2}$ and $-1 \leq B < A \leq 1$. Note here, when $A = 2\lambda - 1$, $0 \leq \lambda \leq 1$, and $B = -1$, (7) reduced to (6).

An Alexander-type equivalence of $\mathcal{S}^\mu[A, B]$ is the set of all functions in $\mathcal{A}$ that satisfies

$$e^{i\mu}\frac{(zf'(z))'}{f'(z)} \prec (\cos\mu)\left(\frac{1+Az}{1+Bz}\right) + i\sin\mu, \qquad z \in \mathbb{U}.$$

This class is donated by $\mathcal{K}^\mu[A, B]$. For more on classes of spiralike functions, see [20].

2 Convoluted Differential Operators

A large number of differential operators have been created. Undoubtedly, many of them are generalised ones. However, still few of them have been combined. In like manner, it is worth to mention some early created differential operators and their power series expansion for forthcoming constructions.

In [19] Ruscheweyh defined the differential operator

$$R^\alpha : \mathcal{A} \to \mathcal{A} \tag{8}$$

where $\alpha \in \mathbb{N}_0 (\mathbb{N}_0 = \{0, 1, 2, \ldots\})$ and

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= zf'(z) \\ R^2 f(z) &= zf'(z) + \frac{1}{2}z^2 f''(z) \\ &\vdots \\ (\alpha+1)R^{\alpha+1} f(z) &= \alpha R^\alpha f(z) + z(R^\alpha f(z))'. \end{aligned}$$

If f is an analytic function of the form (1), then

$$R^\alpha f(z) = \sum_{k=2}^{\infty} C(\alpha, k) a_k z^k, \tag{9}$$

where $C(\alpha, k) = \binom{k+\alpha-1}{\alpha}$.

In [21] Sălăgean defined the following differential operator

$$S^n : \mathcal{A} \to \mathcal{A} \tag{10}$$

where $n \in \mathbb{N}_0$ and

$$S^0 f(z) = f(z)$$

$$S^1 f(z) = zf'(z)$$
$$\vdots$$
$$S^n f(z) = z(S^{n-1}f(z))'.$$

If f is an analytic function of the form (1), then

$$S^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k. \tag{11}$$

Later Al-Oboudi [8] introduced a generalisation of Sălăgean operator which defined as follows:

$$D_\lambda^n : \mathcal{A} \to \mathcal{A} \tag{12}$$

where $n \in \mathbb{N}_0, \lambda \geq 0$ and

$$\begin{aligned}
&D^0 f(z) = f(z)\\
&D^1 f(z) = (1-\lambda)f(z) + zf'(z) = D_\lambda(z) = D_\lambda\\
&\qquad\vdots\\
&D_\lambda^n f(z) = D_\lambda(D^{n-1}f(z)).
\end{aligned}$$

If f is an analytic function of the form (1), then

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n a_k z^k. \tag{13}$$

Many other differential operators have been established and generalised, few to mention [1-9]. In fact, some operators can be written in term of convolution. This comes clear from the definition of convolution which allows coefficients to be splitted. Ruscheweyh's operator can can be observed as following:

$$R^\alpha f(z) = f(z) * \frac{z}{(1-z)^{\alpha+1}}, \qquad \alpha > -1,$$

which implies that

$$R^n f(z) = \frac{z\left(z^{n-1}f(z)\right)^{(n)}}{n!} \qquad n = 1, 2, 3, \ldots.$$

The operator D_λ^n defined in (13) can be written as

$$D_\lambda^n f(z) = \underbrace{\varphi(z) * \cdots * \varphi(z)}_{n-times} * f(z)$$

where

$$\varphi(z) = \frac{z}{1-z} + \frac{\lambda z}{(1-z)^2} - \frac{\lambda z}{1-z}.$$

In [15] Darus and Al-Shaqsi introduced the differential operator

$$R^n_{\alpha,\lambda} : \mathcal{A} \to \mathcal{A}$$

where n, $\alpha \in \mathbb{N}_0$, $\lambda \geq 0$, and

$$\begin{aligned} R^0_{\alpha,\lambda} &= f(z) \\ R^1_{\alpha,\lambda} &= zf'(z) + \lambda z^2 f''(z) = R^* \\ &\vdots \\ R^n_{\alpha,\lambda} f(z) &= R^*\left(R^{n-1}_{\alpha,\lambda} f(z)\right). \end{aligned}$$

If f is an analytic function of the form (1), then

$$R^n_{\alpha,\lambda} f(z) = z + \sum_{k=2}^{\infty} [1 + \beta(k-1)]^n C(\alpha, k)\ a_k z^k. \tag{14}$$

Again, $R^n_{\alpha,\lambda}$ can be rewritten in term of convolution as

$$R^n_{\alpha,\lambda} = \underbrace{\varphi(z) * \cdots * \varphi(z)}_{n-times} * \frac{z}{(1-z)^{\alpha+1}} * f(z).$$

In [18] Lupas considered the differential operator SR^n_α which is the convolution of S^n and R^α. More precisely,

$$\begin{aligned} SR^n_\alpha f(z) &= S^n f(z) * R^\alpha f(z) \\ &= \left(z + \sum_{k=2}^{\infty} k^n a_k z^k\right) * \left(z + \sum_{k=2}^{\infty} C(\alpha, k)\ a_k z^k\right) \\ &= z + \sum_{k=2}^{\infty} k^n C(\alpha, k)\ a_k^2 z^k. \end{aligned}$$

In [11] Andrei considered the differential operator $DR^n_{\alpha,\lambda}$ which is the convolution of D^n_λ and R^α. More precisely,

$$\begin{aligned} DR^n_{\alpha,\lambda} f(z) &= D^n_\lambda f(z) * R^\alpha f(z) \\ &= \left(z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n\ a_k z^k\right) * \left(z + \sum_{k=2}^{\infty} C(\alpha, k)\ a_k z^k\right) \\ &= z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n\ C(\alpha, k)\ a_k^2 z^k. \end{aligned}$$

At the first glance, the differential operators $R^n_{\alpha,\lambda}$ and $DR^n_{\alpha,\lambda}$ appeared quite similar but, indeed, $R^n_{\alpha,\lambda}$ and $DR^n_{\alpha,\lambda}$ are different in term of their construction and even in their coefficients. That is, $R^n_{\alpha,\lambda}$ and $DR^n_{\alpha,\lambda}$ are not equivalent in general. Next example will illustrate the difference.

Example 2.1. *The Koebe function $\kappa(z) = z/(1-z)^2$ under the convoluted differential operator $DR^n_{\alpha,\lambda}$ takes the following manner:*

$$\begin{aligned} DR^n_{\alpha,\lambda}\kappa(z) &= D^n_\lambda \kappa(z) * R^\alpha \kappa(z) \\ &= \left(z + \sum_{k=2}^{\infty} [1+\lambda(k-1)]^n \, k^n z^k \right) * \left(z + \sum_{k=2}^{\infty} C(\alpha,k) \; k^n z^k \right) \\ &= z + \sum_{k=2}^{\infty} [1+\lambda(k-1)]^n \, C(\alpha,k) \; k^{2n} z^k. \end{aligned}$$

On the other hand, $\kappa(z)$ under the differential operator $R^n_{\alpha,\lambda}$ takes different manner as follows:

$$\begin{aligned} R^0_{\alpha,\lambda}\kappa(z) &= \kappa(z) \\ R^1_{\alpha,\lambda}\kappa(z) &= z\kappa'(z) + \lambda z^2 \kappa''(z) = R^* \\ &\vdots \\ R^n_{\alpha,\lambda}\kappa(z) &= R^* \left(R^{n-1}_{\alpha,\lambda}\kappa(z) \right) \\ &= z + \sum_{k=2}^{\infty} [1+\lambda(k-1)]^n \, C(\alpha,k) \; z^k. \end{aligned}$$

Nevertheless, the differential operators $R^n_{\alpha,\lambda}$ and $DR^n_{\alpha,\lambda}$ can be mapped onto same function in some occasions. Consider $\ell(z) = z/(1-z)$. It is clear that

$$R^n_{\alpha,\lambda}\ell(z) = DR^n_{\alpha,\lambda}\ell(z) = z + \sum_{k=2}^{\infty} [1+\lambda(k-1)]^n \, C(\alpha,k) \; z^k.$$

In [7] another convoluted differential operator was constructed by considering the differential operators $R^n_{\alpha,\lambda}$ and D^n_λ. The convoluted operator of both of them is

$$\begin{aligned} \tilde{D}^n_{\alpha,\lambda} f(z) &= D^n_\lambda f(z) * R^n_{\alpha,\lambda} f(z) \\ &= \left(z + \sum_{k=2}^{\infty} [1+\lambda(k-1)]^n \, a_k z^k \right) * \left(z + \sum_{k=2}^{\infty} [1+\lambda(k-1)]^n \, C(\alpha,k) \; a_k z^k \right) \\ &= z + \sum_{k=2}^{\infty} [1+\lambda(k-1)]^{2n} \, C(\alpha,k) \; a_k^2 z^k. \end{aligned}$$

To this end, the way is paved for the main results.

3 Main Results

In this paper, we use the convoluted differential operator $\tilde{D}^n_{\alpha,\lambda}$ to introduce new subclasses of spiralike functions. These new subclasses are given in the next definition.

Definition 3.1. We denote by $\mathcal{S}^{n,\mu}_{\alpha,\lambda}[A,B]$ and $\mathcal{K}^{n,\mu}_{\alpha,\lambda}[A,B]$ the subclasses of $\mathcal{A}$ which are defined by

$$\mathcal{S}^{n,\mu}_{\alpha,\lambda}[A,B]=\left\{f\in\mathcal{A}: e^{i\mu}\frac{z\tilde{D}^n_{\alpha,\lambda}f'(z)}{\tilde{D}^n_{\alpha,\lambda}f(z)}\prec(\cos\mu)\left(\frac{1+Az}{1+Bz}\right)+i\sin\mu\right\},$$

and

$$\mathcal{K}^{n,\mu}_{\alpha,\lambda}[A,B]=\left\{f\in\mathcal{A}: e^{i\mu}\frac{(z\tilde{D}^n_{\alpha,\lambda}f'(z))'}{\tilde{D}^n_{\alpha,\lambda}f'(z)}\prec(\cos\mu)\left(\frac{1+Az}{1+Bz}\right)+i\sin\mu\right\},$$

where $\lambda\geq 0$, and $n,\alpha\in\mathbb{N}_0$.

In the following two subsections, convolution properties and coefficient bounds of the subclasses $\mathcal{S}^{n,\mu}_{\alpha,\lambda}[A,B]$ and $\mathcal{K}^{n,\mu}_{\alpha,\lambda}[A,B]$ are obtained.

3.1 Convolution Properties

We begin with two lemmas due to Bhoosnurnath and Devadas (see [13] and [12]).

Lemma 3.1. *A function f in the class $\mathcal{A}$ is in the class $\mathcal{S}^{\mu}[A,B]$ if and only if*

$$\frac{1}{z}\left[f(z)*(1-Mz)\frac{z}{(1-z)^2}\right]\neq 0,$$

where

$$M=\frac{e^{i\mu}+(A\cos\mu+iB\sin\mu)\zeta}{(A-B)\zeta\cos\mu}. \tag{15}$$

Lemma 3.2. *A function f in the class $\mathcal{A}$ is in the class $\mathcal{K}^{\mu}[A,B]$ if and only if*

$$\frac{1}{z}\left[f(z)*(1-Mz)\frac{z}{(1-z)^3}\right]\neq 0,$$

where

$$N=\frac{2e^{i\mu}+[(A+B)\cos\mu+i2B\sin\mu)]\zeta}{(A-B)\zeta\cos\mu}. \tag{16}$$

The following theorem provides necessary and sufficient condition for the subclass $\mathcal{S}^{n,\mu}_{\alpha,\lambda}[A,B]$.

Theorem 3.1. *A necessary and sufficient condition of a function f in the class $\mathcal{A}$ to be in the subclass $\mathcal{S}^{n,\mu}_{\alpha,\lambda}[A,B]$ is that*

$$1+\sum_{k=2}^{\infty}[(1-M)(k-1)+1]\,[1+\lambda(k-1)]^{2n}\,C(\alpha,k)z^{k-1}\neq 0,$$

where M is given by (15).

Proof. In view of Lemma 3.1, we may write $f\in\mathcal{S}^{n,\mu}_{\alpha,\lambda}[A,B]$ as

$$\frac{1}{z}\left[\tilde{D}^{n}_{\alpha,\lambda}f(z)*\left(\frac{z}{(1-z)^2}-\frac{Mz^2}{(1-z)^2}\right)\right]$$

using (2) and (3) we obtain that

$$\frac{1}{z}\left[\tilde{D}^{n}_{\alpha,\lambda}f(z)*\left(\frac{z}{(1-z)^2}-\frac{Mz^2}{(1-z)^2}\right)\right]$$

$$=\frac{1}{z}\left[z(\tilde{D}^{n}_{\alpha,\lambda}f(z))'-M\left\{z(\tilde{D}^{n}_{\alpha,\lambda}f(z))'-\tilde{D}^{n}_{\alpha,\lambda}f(z)\right\}\right]$$
$$=1+\sum_{k=2}^{\infty}[(1-M)(k-1)+1]\,[1+\lambda(k-1)]^{2n}\,C(\alpha,k)z^{k-1}.$$

This completes the proof. □

The following theorem provides necessary and sufficient condition for the subclass $\mathcal{K}^{n,\mu}_{\alpha,\lambda}[A,B]$.

Theorem 3.2. *A necessary and sufficient condition of a function f in the class $\mathcal{A}$ to be in the subclass $\mathcal{K}^{n,\mu}_{\alpha,\lambda}[A,B]$ is that*

$$1+\sum_{k=2}^{\infty}\frac{(1-N)(k^2+k)}{2}\,[1+\lambda(k-1)]^{2n}\,C(\alpha,k)z^{k-1}\neq 0,$$

where N is given by (16).

Proof. In view of Lemma 3.2, we may write $f\in\mathcal{K}^{n,\mu}_{\alpha,\lambda}[A,B]$ as

$$\frac{1}{z}\left[\tilde{D}^{n}_{\alpha,\lambda}f(z)*\left(\frac{z}{(1-z)^3}-\frac{Mz^2}{(1-z)^3}\right)\right]$$

using (4) and (5) we obtain that

$$\frac{1}{z}\left[\tilde{D}^n_{\alpha,\lambda}f(z) * \left(\frac{z}{(1-z)^3} - \frac{Mz^2}{(1-z)^3}\right)\right]$$

$$= \frac{1}{z}\left[\frac{z}{2}(z\tilde{D}^n_{\alpha,\lambda}f(z))'' - N\left\{\frac{z}{2}(\tilde{D}^n_{\alpha,\lambda}f(z))'' - z(\tilde{D}^n_{\alpha,\lambda}f(z))'\right\}\right]$$
$$= 1 + \sum_{k=2}^{\infty}\frac{(1-N)(k^2+k)}{2}\left[1+\lambda(k-1)\right]^{2n}C(\alpha,k)z^{k-1}.$$

This completes the proof. □

3.2 Coefficient Bounds

In this subsection, coefficient bounds of the subclasses $\mathcal{S}^{n,\mu}_{\alpha,\lambda}[A,B]$ and $\mathcal{K}^{n,\mu}_{\alpha,\lambda}[A,B]$ are obtained.

Theorem 3.3. *If f in the class $\mathcal{A}$ and belongs to the subclass $\mathcal{S}^{n,\mu}_{\alpha,\lambda}[A,B]$, then*

$$\sum_{k=2}^{\infty}\left|\left[(1-M)(k-1)+1\right]\left[1+\lambda(k-1)\right]^{2n}C(\alpha,k)\right| < 1, \tag{17}$$

where M is given (15).

Proof. Since $f \in \mathcal{S}^{n,\mu}_{\alpha,\lambda}[A,B]$, we have

$$\left|1+\sum_{k=2}^{\infty}\left[(1-M)(k-1)+1\right]\left[1+\lambda(k-1)\right]^{2n}C(\alpha,k)z^{k-1}\right| > 0.$$

And since,

$$\left|1+\sum_{k=2}^{\infty}\left[(1-M)(k-1)+1\right]\left[1+\lambda(k-1)\right]^{2n}C(\alpha,k)z^{k-1}\right|$$

$$\geq 1 - \sum_{k=2}^{\infty}\left|\left[(1-M)(k-1)+1\right]\left[1+\lambda(k-1)\right]^{2n}C(\alpha,k)\right| > 0,$$

then

$$\sum_{k=2}^{\infty}\left|\left[(1-M)(k-1)+1\right]\left[1+\lambda(k-1)\right]^{2n}C(\alpha,k)\right| < 1.$$

Therefore, $f(z) \in \mathcal{S}^{n,\mu}_{\alpha,\lambda}[A,B]$. □

The coefficient bounds of the subclass $\mathcal{K}_{\alpha,\lambda}^{n,\mu}[A,B]$ is provided in the next theorem.

Theorem 3.4. *If f in the class $\mathcal{A}$ and belongs to the subclass $\mathcal{K}_{\alpha,\lambda}^{n,\mu}[A,B]$, then*

$$\sum_{k=2}^{\infty}\left|\frac{(1-N)(k^2+k)}{2}\left[1+\lambda(k-1)\right]^{2n}C(\alpha,k)\right|<2, \tag{18}$$

where N is given by (16).

Proof. Since $f\in\mathcal{K}_{\alpha,\lambda}^{n,\mu}[A,B]$, we have

$$\left|1+\sum_{k=2}^{\infty}\frac{(1-N)(k^2+k)}{2}\left[1+\lambda(k-1)\right]^{2n}C(\alpha,k)z^{k-1}\right|>0.$$

And since,

$$\left|1+\sum_{k=2}^{\infty}\frac{(1-N)(k^2+k)}{2}\left[1+\lambda(k-1)\right]^{2n}C(\alpha,k)z^{k-1}\right|$$

$$\geq 1-\sum_{k=2}^{\infty}\left|\frac{(1-N)(k^2+k)}{2}\left[1+\lambda(k-1)\right]^{2n}C(\alpha,k)\right|>0,$$

then

$$\sum_{k=2}^{\infty}\left|\frac{(1-N)(k^2+k)}{2}\left[1+\lambda(k-1)\right]^{2n}C(\alpha,k)\right|<2.$$

Therefore, $f(z)\in\mathcal{K}_{\alpha,\lambda}^{n,\mu}[A,B]$. □

4 Future Work

Verify whether the subclasses $\mathcal{S}_{\alpha,\lambda}^{n,\mu}[A,B]$ and $\mathcal{K}_{\alpha,\lambda}^{n,\mu}[A,B]$ satisfy the inclusion property; that is, is it true that

$$\mathcal{S}_{\alpha,\lambda}^{n+1,\mu}[A,B]\subset\mathcal{S}_{\alpha,\lambda}^{n,\mu}[A,B]$$

and

$$\mathcal{K}_{\alpha,\lambda}^{n+1,\mu}[A,B]\subset\mathcal{K}_{\alpha,\lambda}^{n,\mu}[A,B].$$

Also, for both subclasses, $\mathcal{S}_{\alpha,\lambda}^{n,\mu}[A,B]$ and $\mathcal{K}_{\alpha,\lambda}^{n,\mu}[A,B]$, it is interesting to obtain the upper bound on the functional $|a_3-va_2^2|$ for complex v.

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